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# MECHANICS FOR BEGINNERS.

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WITH NUMEROUS EXAMPLES

BY

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# PREFACE.

THE present work is constructed on the same plan as the author's *Algebra for Beginners* and *Trigonometry for Beginners;* and is intended as a companion to them. It is divided into short Chapters, and a collection of Examples follows each Chapter. Some of these examples are original, and others have been selected from College and University Examination papers.

The work forms an elementary treatise on *demonstrative* mechanics. It may be true that this part of mixed mathematics has been sometimes made too abstract and speculative; but it can hardly be doubted that a knowledge of the elements at least of the theory of the subject is extremely valuable even for those who are mainly concerned with practical results. The author has accordingly endeavoured to provide a suitable introduction to the study of applied as well as of theoretical Mechanics.

The demonstrations will, it is hoped, be found simple and convincing. Great care has been taken to arrange them so as to assume the smallest possible knowledge of pure mathematics, and to furnish the clearest illustration of mechanical principles. At the same time there has been no sacrifice of exactness; so that the beginner may here obtain a solid foundation for his future studies: afterwards he will only have to increase his knowledge without rejecting what he originally acquired. The ex-

perience of teachers shews that it is especially necessary to guard against the introduction of erroneous notions at the commencement of the study of Mechanics.

The work consists of two parts, namely, Statics and Dynamics. It will be found to contain all that is usually comprised in elementary treatises on Mechanics, together with some additions. Thus, for example, an investigation has been given of the time of oscillation of a simple pendulum. The more important cases of central forces are also discussed; partly because they are explicitly required in some examinations, and partly because by the mode of discussion which is adopted they supply valuable exemplifications of fundamental mechanical theorems. It would be easy to treat in the same manner the other cases of central forces which are contained in the first three sections of Newton's Principia. Some notice has been taken of D'Alembert's Principle and of Moment of Inertia, with the view of introducing the reader to the subject of Rigid Dynamics.

As the Chapters of the work are to a great extent independent of each other, it will be possible to vary the order of study at the discretion of the teacher. The Dynamics may with advantage be commenced before the whole of the Statics has been mastered.

I. TODHUNTER.

CAMBRIDGE,
*April*, 1878.

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## STATICS.

## DYNAMICS.

# STATICS.

## I. *Introduction.*

1. We shall commence this work with some preliminary explanations and definitions.

We shall assume that the idea of matter is familiar to the student, being suggested to us by everything which we can touch.

2. A *body* is a portion of matter bounded in every direction. A *material particle* is a body which is indefinitely small in every direction: we shall speak of it for shortness as a particle.

3. Force is that which moves or tends to move a body, or which changes or tends to change the motion of a body.

4. When forces act on a body simultaneously it may happen that they neutralise each other, so as to leave the body at rest. When a body remains at rest, although acted on by forces, it is said to be in *equilibrium.*

5. Mechanics is the science which treats of the equilibrium and motion of bodies. Statics treats of the equilibrium of bodies, and Dynamics of the motion of bodies.

6. Mechanics may be studied to a certain extent as a purely experimental subject; but the knowledge gained in this way will be neither extensive nor sound. To increase the range and to add to the security of our investigations we require the aid of the sciences of space and number.

These sciences form *Pure Mathematics:* Arithmetic and Algebra relate to number, and Geometry to space: Trigonometry is formed by the combination of Arithmetic and Algebra with Geometry. When Mechanics is studied with the aid of Pure Mathematics it is often called Demonstrative Mechanics, or Rational Mechanics, to indicate more distinctly that the results are deduced from principles by exact reasoning. Demonstrative Mechanics constitutes a portion of *Mixed Mathematics.*

7. In the present work we shall give the elements of Demonstrative Mechanics. We shall assume that the student is acquainted with Geometry as contained in Euclid, and with Algebra and Trigonometry so far as they are carried in the Treatises for Beginners.

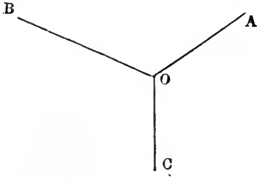
8. There are three things to consider in a force acting on a particle: the *point of application* of the force, that is, the position of the particle on which the force acts; the *direction* of the force, that is, the direction in which it tends to make the particle move; and the *magnitude* of the force.

9. The *position* of a particle may be determined in the same way as the position of a point in Geometry. The *direction* of a force may be determined in the same way as the direction of a straight line in Geometry. We shall proceed to explain how we measure the *magnitude* of a force.

10. Forces may be measured by taking some force as the unit, and expressing by numbers the ratios which other forces bear to the unit. Two forces are *equal* when on being applied in opposite directions to a particle they keep it in equilibrium. If we take two equal forces and apply them to a particle in the same direction we obtain a force *double* of either; if we combine *three* equal forces we obtain a *triple* force; and so on.

11. Thus we may very conveniently represent forces by straight lines. For we may draw a straight line from the point of application of the force in the direction of the force, and of a length proportional to the magnitude of the force.

Thus, suppose a particle acted on by three forces $P$, $Q$, $R$ in three different directions. We may take straight lines to represent these forces. For let $O$ denote the position of a particle. Draw straight lines $OA$, $OB$, $OC$ in the directions of the forces $P$, $Q$, $R$ respectively; and take the lengths of these straight lines proportional to the forces: that is, take

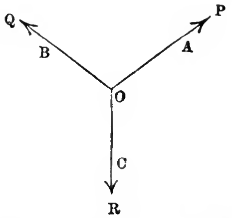



$$\frac{OB}{OA} = \frac{Q}{P} \text{ and } \frac{OC}{OB} = \frac{R}{Q}.$$

In saying that $OA$ represents the force $P$, we suppose that this force acts from $O$ towards $A$; if the force acted from $A$ towards $O$ we should say that $AO$ represents it. This distinction is indeed sometimes neglected, but it may be observed with great advantage.

It would be convenient to distinguish between the phrases *line of action* and *direction:* thus, if we say that $OA$ is the *line of action* of a force $P$, it may be understood that the force merely acts in this straight line, either from $O$ to $A$, or from $A$ to $O$; but if we say that $OA$ is the *direction* of a force $P$ it may be understood that the force acts from $O$ towards $A$.

12. Sometimes an arrow head is used in a figure to indicate which end of the straight line representing the force is that *towards* which the force tends. Sometimes the letters, as $P$, $Q$, $R$, which denote the magnitudes of the forces, are inserted in the figure.




13. We find by experiment that if a body be set free it will fall downwards in a certain direction; if it start

again from the same point as before it will reach the ground at the same point as before. This direction is called the *vertical* direction, and a plane perpendicular to it is called a *horizontal* plane. The cause of this effect is assumed to be a certain power in the earth which is called *attraction*, or sometimes *gravity*.

If the body be prevented from falling by the interposition of a hand or a table, the body exerts a *pressure* on the hand or table.

Weight is the name given to the pressure which the attraction of the earth causes a body to exert on another with which it is in contact.

14. In Statics forces are usually measured by the weights which they can support. Thus, if we denote the force which can support one pound by 1, we denote the force which can support five pounds by 5.

15. Force may be exerted in various ways; but only three will come under our notice:

We may *push* a body by another; for example, by the hand or by a rod: force so exerted may be called *pressure*.

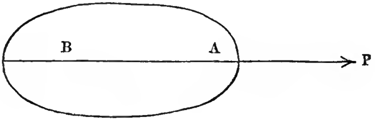
We may *pull* a body by means of a string or of a rod: force exerted by means of a string, or by means of a rod used like a string, may be called *tension*.

The attraction of the earth is exerted *without the intervention of any visible instrument:* it is the only force of the kind with which we shall be concerned.

16. A solid body is conceived to be an aggregation of material particles. A rigid body is one in which the particles retain invariable positions with respect to each other. No body in nature is perfectly rigid; every body yields more or less to the forces which act on it. But investigations as to the way in which the particles of a solid body are held together, and as to the deviation of bodies from perfect rigidity, belong to a very abstruse part of Mechanics. It is found sufficient in elementary works to assume that the rigidity is practically perfect.

17. We shall now enunciate an important principle of Statics which the student must take as an axiom: *when a force acts on a body the effect of the force will be unchanged at whatever point of its direction it be applied, provided this point be a point of the body or be rigidly connected with the body.*

Thus, suppose a body kept in equilibrium by a system of forces, one of which is the force $P$ applied at the point

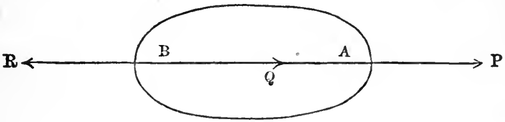



$A$. Let $B$ be any point on the straight line which coincides with the direction of $P$. Then the axiom asserts that $P$ may be applied at $B$ instead of at $A$, and that the effect of the force will remain the same.

This principle is called the *transmissibility of a force to any point in its line of action.* It may be verified to some extent by direct experiment; but the best evidence we have of its truth is of an indirect character. We assume that the principle is true; and we construct on this basis the whole theory of Statics. Then we can compare many of the results which we obtain with observation and experiment; and we find the agreement so close that we may fairly infer that the principle on which the theory rests is true. But we cannot appeal to evidence of this kind at the beginning of the subject, and we therefore take the principle as an axiom the truth of which is to be assumed.

18. The following remarks will give a good idea of the amount of assumption involved in the axiom of the preceding Article.

At the point $B$ suppose we apply two forces $Q$ and $R$, each equal to $P$, the former in the direction of $P$, and the




latter in the opposite direction. Then we may readily admit that we have made no change in the action of $P$.

Now $P$ at $A$ and $R$ at $B$ are equal forces acting in opposite directions; *let us assume* that they neutralise each other: then these two forces may be removed without disturbing the equilibrium of the body, and there will remain the force $Q$ at $B$, that is, a force equal to $P$ and applied at $B$ instead of at $A$.

19. When we find it useful to change the point of application of a force, we shall for shortness not always state that the new point is *rigidly connected* with the old point; but this must be always understood.

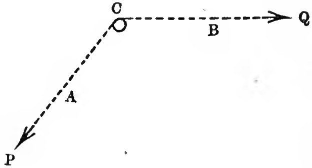
20. We shall have occasion hereafter to assume what may be called the *converse* of the principle of the transmissibility of force, namely, that if a force can be transferred from its point of application to a second point without altering its effect, then the second point must be in the line of action of the force.

21. We shall frequently have to refer to an important property of a string considered as an instrument for exerting force, which we will now explain.

Let $AB$ represent a string pulled at one end by a force $P$, and at the other end by a force $Q$, in opposite directions.


It is clear that if the string is in equilibrium the forces must be equal.

If the force $Q$ be applied at any intermediate point $C$, instead of at $B$, still for equilibrium we must have $Q$ equal to $P$. This is sometimes expressed by saying that *force transmitted directly by a string is transmitted without change.*




Again, let a string $ACB$ be stretched round a smooth peg $C$; then we may admit that if the string be in equilibrium the forces $P$ and $Q$ at its ends must be equal. This is sometimes expressed by saying that *force transmitted by a string round a smooth peg is transmitted without change.*

Or in both cases we may say briefly, that *the tension of the string is the same throughout.*

We suppose that the weight of the string itself may be left out of consideration.

22. Experiment shews that the weight of a certain volume of one substance is not necessarily the same as the weight of an equal volume of another substance. Thus, 7 cubic inches of iron weigh about as much as 5 cubic inches of lead. We say then that lead is *denser* than iron; and we adopt the following definitions:

When the weight of any portion of a body is proportional to the volume of that portion the body is said to be of *uniform density.* And the densities of two bodies of uniform density are proportional to the weights of equal volumes of the bodies. Thus we may take any body of uniform density as the standard and call its density 1, and then the density of any other body will be expressed by a number. For example, suppose we take *water* as the standard substance; then since a cubic inch of copper weighs about as much as 9 cubic inches of water, the density of copper will be expressed by the number 9.

## Examples. I.

1. If a force which can just sustain a weight of 5 lbs. be represented by a straight line whose length is 1 foot 3 inches, what force will be represented by a straight line 2 feet long?

2. How would a force of a ton be represented if a straight line an inch long were the representation of a force of seventy pounds?

3. If a force of $P$ lbs. be represented by a straight line $a$ inches long, what force will be represented by a straight line $b$ inches long?

4. If a force of $P$ lbs. be represented by a straight line $a$ inches long, by what straight line will a force of $Q$ lbs. be represented?

5. A string suspended from a ceiling supports a weight of 3 lbs. at its extremity, and a weight of 6 lbs. at its middle point: find the tensions of the two parts of the string. If the tension of the upper part be represented by a straight line 3 inches long, what must be the length of the straight line which will represent the tension of the lower portion?

6. If 3 lbs. of brass are as large as 4 lbs. of lead, compare the densities of brass and lead.

7. Compare the densities of two substances $A$ and $B$ when the weight of 3 cubic inches of $A$ is equal to the weight of 4 cubic inches of $B$.

8. A cubic foot of a substance weighs 4 cwt.: what bulk of another substance five times as dense will weigh 7 cwt.?

9. Two bodies whose volumes are as 3 is to 4 are in weight as 4 is to 3: compare their densities.

10. If the weight of $a$ cubic inches of one substance and of $b$ cubic inches of another substance be in the ratio of $m$ to $n$, compare the densities of the substances.

## II. *Parallelogram of Forces.*

23. When two forces act on a particle and do not keep it in equilibrium, the particle will begin to move in some definite direction. It is clear then that a *single* force may be found such that if it acted in the direction opposite to that in which the motion would take place, this force would prevent the motion, and consequently would be in equilibrium with the other forces which act on the particle. If then we were to remove the original forces, and replace them by a single force equal in magnitude to that just considered, but acting in the opposite direction, the particle would still be in equilibrium.

Hence we are naturally led to adopt the following definitions:

A force which is equivalent in effect to two or more forces is called their *resultant;* and these forces are called *components* of the resultant.

24. We have then to consider the *composition of forces*, that is, the method of finding the resultant of two or more forces. The present Chapter will be devoted to the case of *two* forces acting on a particle.

25. *When two forces act on a particle in the same direction their resultant is equal to their sum and acts in the same direction.*

This is obvious. For example, if a force of 5 lbs. and a force of 3 lbs. act on a particle in the same direction their resultant is a force of 8 lbs., acting in the same direction.

26. *When two forces act on a particle in opposite directions their resultant is equal to their difference, and acts in the direction of the greater force.*

This is obvious. For example, if a force of 5 lbs. and a force of 3 lbs. act on a particle in opposite directions their resultant is a force of 2 lbs., acting in the same direction as the force of 5 lbs.

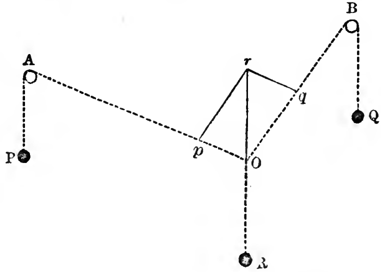
27. We must now proceed to the case in which two forces act on a particle in directions which do not lie in the same straight line; the resultant is then determined by the following proposition:

*If two forces acting on a particle be represented in magnitude and direction by straight lines drawn from a point, and a parallelogram be constructed having these straight lines as adjacent sides, then the resultant of the two forces is represented in magnitude and direction by that diagonal of the parallelogram which passes through the point.*

This proposition is the most important in the science of Statics; it is called briefly the *Parallelogram of Forces.* We shall first shew how the proposition may be verified experimentally; we shall next point out various interesting results to which it leads; and finally demonstrate it.

28. *To verify the Parallelogram of Forces experimentally.*

Let $A$ and $B$ be smooth horizontal pegs fixed in a vertical wall. Let three strings be knotted together; let




$O$ represent the knot. Let one string pass over the peg $A$ and have a weight $P$ attached to its end; let another string pass over the peg $B$ and have a weight $Q$ attached to its end; let the other string hang from $O$ and have a weight $R$ attached to its end: any two of these three weights must be greater than the third. Let the system be allowed to adjust itself so as to be at rest. By Art. 21 the pegs do not change the effects of the weights $P$ and $Q$ as to magnitude.

We have three forces acting on the knot at $O$, and

keeping it in equilibrium; so that the effect of $P$ along $OA$ and of $Q$ along $OB$ is just balanced by the effect of $R$ acting vertically downwards at $O$. Therefore the *resultant* of $P$ along $OA$ and of $Q$ along $OB$ must be a force equal to $R$, acting vertically upwards at $O$.

Now on $OA$ take $Op$ to contain as many inches as the weight $P$ contains pounds; and on $OB$ take $Oq$ to contain as many inches as the weight $Q$ contains pounds; and complete the parallelogram $Oqrp$. Then it will be found by trial that $Or$ contains as many inches as the weight $R$ contains pounds; and that $Or$ is a vertical line.

We may change the positions of the pegs, and the magnitudes of the weights employed, in order to give due variety to the experiment; and the general results afford sufficient evidence of the truth of the *Parallelogram of Forces.* The strings should be fine and very flexible in order to promote the success of the experiment; and it is found practically that small pullies serve better than fixed pegs for changing the *directions* of action of the weights $P$ and $Q$ without changing the *amounts* of action.

We proceed now to the numerical calculation of the resultant of two forces.

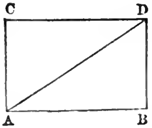
29. The case in which two forces act on a particle in directions which include a right angle deserves especial notice.

Let $AB$ represent a force $P$, and $AC$ a force $Q$; and let $BAC$ be a right angle. Complete the rectangle $ABDC$: then $AD$ represents the resultant of $P$ and $Q$; we will denote this resultant by $R$.

Now by Euclid I. 47,

$$AD^2 = AB^2 + BD^2 = AB^2 + AC^2;$$

so that $$R^2 = P^2 + Q^2.$$



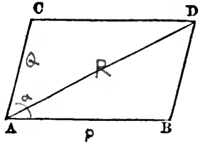

For example, let $P$ be 15 lbs., and $Q$ be 8 lbs.: then

$$R^2 = (15)^2 + (8)^2 = 225 + 64 = 289;$$

therefore $$R = 17.$$

Thus the resultant force is 17 lbs.

30. We will now give the general expression for the resultant of two forces which act on a particle whatever be the angle between their directions.




Let $AB$ represent a force $P$, and $AC$ a force $Q$; and let $a$ denote the angle $BAC$. Complete the parallelogram $ABDC$: then $AD$ represents the resultant of $P$ and $Q$; we will denote the resultant by $R$.

Now by Trigonometry,

$$AD^2 = AB^2 + AC^2 + 2AB \,.\, AC \cos BAC,$$

so that $$R^2 = P^2 + Q^2 + 2PQ \cos a.$$

For example, let $Q$ be equal to $P$, and let $a$ be $60^0$, then $\cos 60^0 = \frac{1}{2}$, so that

$$R^2 = P^2 + P^2 + P^2 = 3P^2,$$

therefore $$R = P\sqrt{3}.$$

Whatever be the angle $a$ if $P = Q$ we have

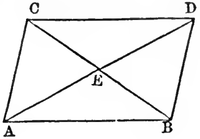
$$R^2 = P^2 + P^2 + 2P^2 \cos a = 2P^2 (1 + \cos a);$$

but $$1 + \cos a = 2\cos^2 \frac{a}{2};$$

therefore $$R^2 = 4P^2 \cos^2 \frac{a}{2}, \text{ and } R = 2P \cos \frac{a}{2}.$$

31. Let $AB$ and $AC$ represent two forces; and $AD$ their resultant. Draw $CB$ the other diagonal of the parallelogram.

Then since the diagonals of a parallelogram bisect each other, $CE = EB$, and $AD = 2AE$. Hence the resultant of the two forces may be determined thus: join $CB$ and bisect it at $E$; then $AE$ is the direction of the resultant, and the magnitude of the resultant is twice $AE$. This mode of determining the resultant is often useful.



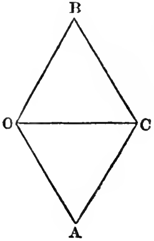

32. Suppose three equal forces to act on a particle, and the direction of each to make an angle of $120^0$ with the directions of the two others: see the diagram in Art. 12. Then it is obvious that the particle will be in equilibrium, for there is no reason why it should move in one direction rather than in another.

This result is in accordance with the Parallelogram of Forces.

For let $OA$ and $OB$ be equal straight lines; and let the angle $AOB$ be $120^0$. Complete the parallelogram $OACB$.

Then $AC = OB = OA$; and the angle $OAC$ is $60^0$; therefore the angle $ACO$ = the angle $AOC = 60^0$. Hence the triangle $OAC$ is equilateral, so that $OC = OA$.

Thus the resultant of two equal forces, the directions of which include an angle of $120^0$, is equal to either of the forces, and bisects the angle between them.

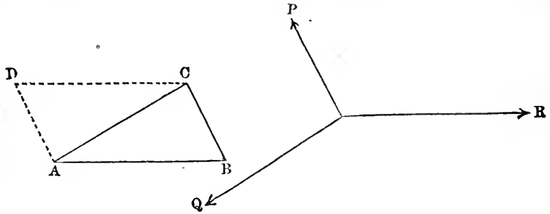



33. Suppose that three forces act on a particle and keep it in equilibrium, and that two of the forces are equal: then the third force must be equally inclined to the directions of the other two.

Hence, if the resultant of two forces is equal in magnitude to one of the forces, the other force is at right angles to the straight line which bisects the angle between the resultant and the force to which it is equal.

34. *If three forces acting on a particle be represented in magnitude and way of action by the sides of a triangle taken in order they will keep the particle in equilibrium.*

Let $ABC$ be a triangle; let $P$, $Q$, $R$ be three forces proportional to the sides $BC$, $CA$, $AB$; let these forces



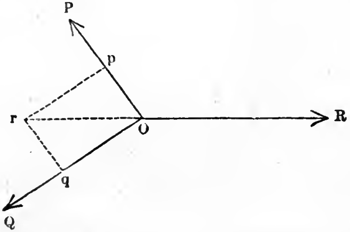

act on a particle, $P$ parallel to $BC$, $Q$ parallel to $CA$, and $R$ parallel to $AB$: then the particle will be in equilibrium.

For draw $AD$ parallel to $BC$, and $CD$ parallel to $BA$. Forces represented by $AB$ and $AD$ in magnitude and direction will have a resultant represented by $AC$ in magnitude and direction. Therefore forces represented by $AB$, $AD$, and $CA$ in magnitude and direction will be in equilibrium; and $AD$ is equal and parallel to $BC$. Thus the proposition is true.

35. The preceding proposition is usually called the *Triangle of Forces.* The student should pay careful attention to the enunciation, in order to understand distinctly what is here asserted. The words *taken in order* must be noticed. If one force is represented by $AB$ the others must be represented by $BC$ and $CA$, not by $BC$ and $AC$, nor by $CB$ and $CA$, nor by $CB$ and $AC$. Also, it is to be observed that the forces are supposed to act *on a particle,* that is, to have a common point of application. Thus the directions of the forces are not represented strictly by $AB$, $BC$, and $CA$, but by straight lines parallel to these drawn *from a common point.* Beginners frequently make a mistake in this respect; they imagine that forces, actually represented in magnitude and situation by $AB$, $BC$, and

$CA$, would keep a body in equilibrium: that is, they forget the limitation that the forces *must act on a particle.* In order to direct the attention of the student to this important limitation we have employed the words *way of action* in the enunciation, instead of the usual word *direction*, which is liable to be confounded with *situation.*

36. *If three forces acting on a particle keep it in equilibrium, and a triangle be drawn having its sides parallel to the lines of action of the forces, the sides of the triangle will be proportional to the forces respectively parallel to them.*




Let forces $P$, $Q$, $R$ acting on a particle at $O$ keep it in equilibrium. In the direction of $P$ take any point $p$, and in the direction of $Q$ take a point $q$, such that $\dfrac{Oq}{Op} = \dfrac{Q}{P}$. Complete the parallelogram $Oprq$. Then $Or$ represents the resultant of $P$ and $Q$ in magnitude and direction. Hence $r$ must be on the straight line $RO$ produced. Therefore any triangle having its sides parallel to the directions of the forces will be similar to the triangle $Opr$, and will therefore have its sides proportional to the forces in magnitude.

37. *If three forces acting on a particle keep it in equilibrium, and a triangle be drawn having its sides, or its sides produced, perpendicular to the lines of action of the forces, the sides of the triangle will be proportional to the forces respectively perpendicular to them.*

For suppose $ABC$ to denote any triangle. Then straight

lines perpendicular respectively to $AB$ and $AC$ will include an angle equal to $A$; and so on. Thus the triangle which has its sides perpendicular to those of $ABC$ will be equiangular to $ABC$, and therefore similar to it. The side which is perpendicular to $BC$ will be opposite to an angle equal to $A$; and so on. Now by Art. 36 the forces will be represented by the sides of a triangle *parallel* to the lines of action; and therefore they will also be represented by the sides of a triangle *perpendicular* to the lines of action.

38. *If three forces acting on a particle keep it in equilibrium, each force is proportional to the sine of the angle between the directions of the other two.*

Let forces $P$, $Q$, $R$ acting on a particle at $O$ keep it in equilibrium. Then, as in Art. 36, we have

$$P : Q : R = Op : pr : rO.$$

But, by Trigonometry,

$$\begin{aligned} Op : pr : rO &= \sin prO : \sin pOr : \sin rpO \\ &= \sin QOR : \sin ROP : \sin POQ. \end{aligned}$$

39. Conversely, if three forces act on a particle, and each force is proportional to the sine of the angle between the other two, the forces will keep the particle in equilibrium provided a certain condition be fulfilled. This condition corresponds to that expressed by the words *taken in order* of Arts. 34 and 35. Thus if $Op$ and $Oq$ in the diagram of Art. 36 represent the directions of two of the forces, the third force must be in the direction $rO$, and not in the direction $Or$. We may express this by saying that the direction of the third force must lie outside the angle formed by the directions of the other two forces; the word *direction* being taken strictly as in Art. 11.

Similarly the converse of Art. 37 is true, provided this condition holds.

40. If two forces act *on a particle* their effect is equivalent to that of a third force of suitable magnitude and direction; if the two forces act *on a body at a point*, we may take it as obvious that their effect is also equivalent to that of

the same third force acting on the body at the point. Hence results obtained with respect to forces acting *on a particle* may be applied with respect to forces acting *on a body at a point:* and in stating such results we often, for brevity, omit the words *on a body* and speak of forces acting *at a point.* Thus the important enunciation of Art. 27 may be modified by changing the words *on a particle* to *at a point.*

If *three* forces act at a point we may find their resultant thus: form the resultant of two of them, then form the resultant of this and the third force. Thus we have the resultant of the three forces. This process may be extended to more than three forces; we shall consider it more fully in the next Chapter.

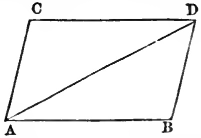
41. *If three forces acting in one plane maintain a rigid body in equilibrium their lines of action either all meet at a point or are all parallel.*

For suppose the lines of action of two of the forces to meet at a point; these forces may be supposed to act at this point, and may be replaced by their resultant. This resultant and the third force must then be equal and opposite in order to maintain the body in equilibrium: so that the line of action of the third force must pass through the intersection of the lines of action of the other two. Then the forces must satisfy the conditions of Art. 36.

The conditions which must hold among three parallel forces which keep a rigid body in equilibrium will be given in Chapter IV.

42. As we can *compound* two forces into one, so on the other hand we can *resolve* one force into two others.

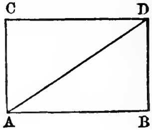
For let $AD$ represent a force; draw any parallelogram $ABDC$ having $AD$ as a diagonal; then the force represented by $AD$ is equivalent to two forces represented by $AB$ and $AC$ respectively.




Thus we can resolve any force into two components which shall have *assigned* directions.

The case in which we resolve a force into two others at right angles deserves special notice.

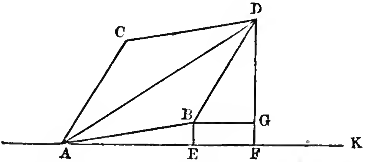
Let $BAC$ be a right angle; and let $a$ denote the angle $DAB$. Then $AB=AD\cos a$, $AC=BD=AD\sin a$. Thus any force $P$ may be resolved into two others, $P\cos a$ and $P\sin a$, which are at right angles: the direction of the component $P\cos a$ making an angle $a$ with the direction of $P$.




43. We see from the former part of the preceding Article that a given force may be resolved into two others in an infinite number of ways. In future when we speak of the resolved part of a force in a given direction we shall always suppose, unless the contrary is expressed, that the force is resolved into two forces, one in the given direction, and the other in the direction at right angles to the given direction; and the former component we shall call the resolved force in the given direction.

44. *The resolved part in any direction of the resultant of two forces acting at a point is equal to the sum of the resolved parts of the components in that direction.*

Let $AB$ and $AC$ denote two forces, and $AD$ their resultant.




Let $AK$ be any straight line through $A$. Draw $BE$ and $DF$ perpendicular to $AK$, and $BG$ parallel to $AK$.

Then $AF$ represents the resolved part of the resultant along $AK$; and $AF=AE+EF$. Now $AE$ is the resolved

part of the component $AB$ along $AK$; and $EF$ is equal to $BG$, that is to the resolved part along $AK$ of the component $AC$; for $BD$ is equal and parallel to $AC$.

45. We shall now give the *demonstration* of the truth of the Parallelogram of Forces which is most suitable for an elementary work; it is called by the name of Duchayla, to whom it is due.

The demonstration is divided into three parts: in the first part the proposition is established so far as relates to the *direction* of the resultant, the forces being commensurable; in the second part this result is extended to incommensurable forces; in the third part the proposition is established with respect to the *magnitude* of the resultant.

46. We have first to notice a preliminary assumption.

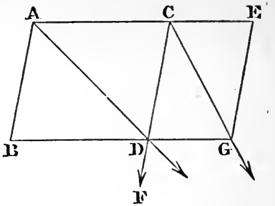
We assume that if two *equal* forces act on a particle the direction of the resultant will be in the same plane as the directions of the forces and will bisect the angle between them. This seems obvious, for there is no reason why the resultant should lie on one side of the plane of the forces rather than on the other; and there is no reason why it should be nearer to one force than to the other.

If a parallelogram be constructed on two equal straight lines meeting at a point as adjacent sides, the diagonal which passes through that point bisects the angle between the sides which meet at that point.

Hence the parallelogram of forces is true, so far as the direction is concerned, when the forces are *equal*.

47. *To demonstrate the Parallelogram of Forces so far as relates to the direction of the resultant, the forces being commensurable.*

Suppose that the proposition is true for two forces $P$ and $Q$, inclined at any angle; and also for two forces $P$ and $R$ inclined at the same angle: we shall shew that it will be true for two forces $P$ and $Q+R$ inclined at the same angle.




Let $A$ be the point of application of the forces; let the force $P$ act along $AB$ and the force $Q + R$ along $AE$.

Let $AB$ and $AC$ represent $P$ and $Q$ in magnitude, and let $CE$ represent $R$ in magnitude. Complete the parallelogram $ACDB$, and the parallelogram $CEGD$.

By Art. 17 the force $R$ may be supposed to act at $C$ instead of at $A$; and thus may be denoted in magnitude and situation by $CE$.

Now by hypothesis the resultant of $P$ and $Q$ acts along $AD$; let $P$ and $Q$ be replaced by their resultant, and let this resultant be supposed to act at $D$ instead of at $A$. Resolve this force at $D$ into two components, one along $CD$ and the other along $DG$: these two components will be respectively $P$ and $Q$, the former may be supposed to act at $C$, and the latter to act at $G$.

Again, the resultant of $P$ along $CD$ and $R$ along $CE$ acts by hypothesis along $CG$; let $P$ and $R$ be replaced by their resultant, and let this resultant be supposed to act at $G$.

Thus by this process we have transferred the forces which acted at $A$ to $G$, without altering their effect. Hence, by Art. 20, we infer that $G$ is a point in the direction of the resultant of the forces $P$ and $Q+R$ at $A$; that is, the resultant of $P$ and $Q+R$ acts in the direction of the diagonal $AG$; thus, if the proposition hold for $P$ and $Q$ and for $P$ and $R$, it holds for $P$ and $Q+R$. But the proposition holds for equal forces $P$ and $P$, therefore it holds for $P$ and $2P$, and therefore for $P$ and $3P$, and so on; hence it holds for $P$ and $nP$, where $n$ is any whole number.

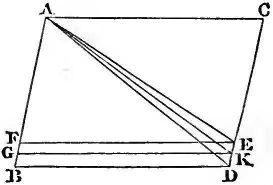
And since it holds for $P$ and $nP$, it holds for $2P$ and $nP$, and therefore for $3P$ and $nP$, and so on; hence it holds for $mP$ and $nP$, where $m$ is any whole number.

Thus the proposition holds for any two commensurable forces.

48. *To demonstrate the Parallelogram of Forces so far as relates to the direction of the resultant, the forces being incommensurable.*

The result in this case may be inferred from the fact that when two magnitudes are incommensurable, so that the ratio of the one to the other cannot be expressed *exactly* by means of numbers, we can find numbers which shall represent the ratio within any assigned degree of closeness.

The result may also be established indirectly thus:




Let $AB$, $AC$ represent two incommensurable forces. Complete the parallelogram $BC$. Then if their resultant do not act along $AD$ suppose it to act along $AE$: draw $EF$ parallel to $CA$.

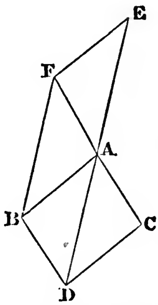
Divide $AC$ into a number of equal portions, each less than $DE$; mark off on $CD$ portions equal to these, and let $K$ be the last division: then $K$ evidently falls between $D$ and $E$. Draw $KG$ parallel to $CA$.

Then two forces represented by $AC$ and $AG$ have a resultant in the direction $AK$, because the forces are commensurable. Therefore the forces $AC$ and $AB$ are equivalent to a force along $AK$, together with a force equal to $GB$ applied at $A$ along $AB$. And we may assume as obvious that the resultant of these forces must lie between $AK$ and $AB$; but by hypothesis the resultant is $AE$, which is *not* between $AK$ and $AB$: this is absurd.

In the same way we may shew that the resultant cannot act along any straight line except $AD$.

Thus the Parallelogram of Forces is demonstrated so far as relates to the *direction* of the resultant whether the forces are commensurable or incommensurable.

49. *To demonstrate that the Parallelogram of Forces holds also with respect to the magnitude of the resultant.*




Let $AB$, $AC$ be the directions of the given forces, $AD$ that of their resultant. Take $AE$ opposite to $AD$, and of such a length as to represent the *magnitude* of the resultant.

Then the forces represented by $AB$, $AC$, $AE$ balance each other. On $AE$ and $AB$ as adjacent sides construct the parallelogram $ABFE$: then the diagonal $AF$ is the direction of the resultant of $AE$ and $AB$.

Hence $AC$ must be in the same straight line with $AF$; therefore $AFBD$ is a parallelogram; therefore $AD=BF$. But $BF=AE$. Therefore $AE=AD$: hence the resultant is represented in *magnitude* as well as in direction by the diagonal $AD$.

Thus the *Parallelogram of Forces* is completely demonstrated.

## Examples. II.

1. Two forces act on a particle, and their greatest and least resultants are 72 lbs. and 56 lbs.: find the forces.

2. Find the resultant of two forces of 12 lbs. and 35 lbs. respectively which act at right angles on a particle.

3. Two forces whose magnitudes are as 3 is to 4, acting on a particle at right angles to each other, produce a resultant of 15 lbs.: find the forces.

4. Two forces, one of which is double of the other, act on a particle, and are such that if 6 lbs. be added to the larger, and the smaller be doubled, the direction of the resultant is unchanged: find the forces.

5. Shew that if the angle at which two given forces are inclined to each other is increased their resultant is diminished.

6. If one of two forces acting on a particle is 5 lbs., and the resultant is also 5 lbs., and at right angles to the known force, find the magnitude and the direction of the other force.

7. If the resultant of two forces is at right angles to one of the forces, shew that it is less than the other force.

8. If the resultant of two forces is at right angles to one force and equal to half the other, compare the forces.

9. If forces of 3 lbs. and 4 lbs. have a resultant of 5 lbs., at what angle do they act?

10. If two forces acting at right angles to each other be in the proportion of 1 to $\sqrt{3}$, and their resultant be 10 lbs., find the forces.

11. How can forces of 43 and 65 lbs. be applied to a particle so that their resultant may be 22 lbs.?

12. Find at what angle the forces $P$ and $2P$ must act at a point in order that the direction of their resultant may be at right angles to the direction of one of the forces.

13. Two strings, the lengths of which are 6 and 8 inches respectively, have their ends fastened at two points the distance between which is 10 inches; their other ends are fastened together, and they are strained tight by a force equivalent to 5 lbs. at the knot acting perpendicularly to the straight line joining the two points: find the tension of each string.

14. $AB$ and $AC$ are adjacent sides of a parallelogram, and $AD$ is a diagonal; $AB$ is bisected at $E$: shew that the resultant of the forces represented by $AD$ and $AC$ is double the resultant of the forces represented by $AE$ and $AC$.

15. Two forces are represented in magnitude and direction by two chords of a circle, drawn from a point on the circumference at right angles to each other: shew that the resultant is represented in magnitude and direction by the diameter which passes through the point.

16. $A$ and $B$ are fixed points; at a point $M$ forces of given magnitude act along $MA$ and $MB$: if their resultant is of constant magnitude, shew that $M$ lies on one or other of two equal arcs described on $AB$ as chord.

## III. *Forces in one plane acting on a particle.*

50. We shall now shew how to determine the resultant of any number of forces in one plane acting on a particle; we have already briefly noticed this subject: see Art 40.

51. *To find the resultant of a given number of forces acting on a particle in the same straight line.*

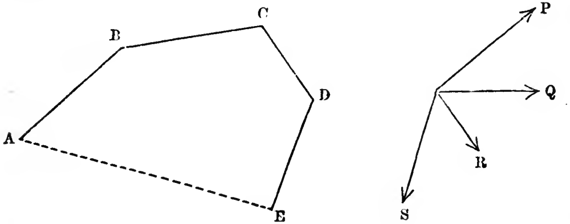
When several forces act on a particle in the *same* direction their resultant is equal to their sum.

When some forces act in *one* direction, and other forces act in the *opposite* direction, the whole force in each direction is the sum of the forces in that direction. Hence the resultant of all the forces is equal to the difference of those two sums, and acts in the direction of the greater sum.

If the forces acting in one direction are reckoned *positive*, and those acting in the other direction *negative*, then the resultant of all the forces is equal to their algebraical sum; and the sign of this sum determines the direction in which the resultant acts.

If the algebraical sum is zero the forces are in equilibrium; and conversely, if the forces are in equilibrium their algebraical sum is zero.

52. *To determine geometrically the resultant of any number of forces acting on a particle.*




Let forces $P$, $Q$, $R$, $S$ act on a particle: it is required to determine their resultant.

Take any point $A$ and draw the straight line $AB$ to represent the force $P$ in magnitude and way of action; from $B$ draw $BC$ to represent $Q$ in magnitude and way of action; from $C$ draw $CD$ to represent $R$ in magnitude and way of action; and from $D$ draw $DE$ to represent $S$ in magnitude and way of action. Join $AE$; then $AE$ will represent the resultant in magnitude and way of action.

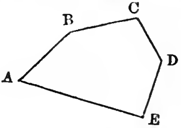
This is obvious from the preceding Chapter. For the resultant of $P$ and $Q$ would be represented by the straight line $AC$; and then the resultant of the forces represented by $AC$ and $CD$ would be represented by $AD$; that is, $AD$ would represent the resultant of $P$, $Q$, and $R$; and so on.

The method is applicable whatever be the number of forces acting on a particle.

Hence we easily see that Art. 44 may be extended to the case of any number of forces.

53. *If any number of forces acting on a particle be represented in magnitude and way of action by the sides of a polygon taken in order, they will keep the particle in equilibrium.*

Take for example a polygon of five sides. Let forces represented in magnitude and way of action by $AB$, $BC$, $CD$, $DE$, $EA$ act on a particle: they will keep the particle in equilibrium.




For, by the preceding Article, the forces represented by $AB$, $BC$, $CD$, $DE$ have a resultant which may be represented by $AE$: and two forces represented by $AE$ and $EA$ respectively will balance each other.

54. The preceding proposition is usually called the *Polygon of Forces.* The remarks made in Art. 35 respecting the *Triangle of Forces* are applicable here also.

The converse of the *Triangle* of Forces is true, as is shewn in Art. 36; but the converse of the *Polygon* of Forces is not true; that is, if four or more forces acting on a particle keep it in equilibrium, we cannot assert that the

forces are proportional to the sides of *any* polygon which has its sides parallel to the lines of action of the forces. For one polygon may be equiangular to another without being similar to it. If in the figure of the preceding Article we draw any straight line parallel to one of the sides, as $AE$ for example, we can form a second polygon, which like the first has its sides parallel to the lines of action of the forces; but the sides of the one polygon are not in the same relative proportion as the sides of the other.

55. It will be seen that the geometrical process of Art. 52 is applicable when the forces do not all lie in one plane. Also in Art. 53 the polygon need not be restricted to be a plane polygon. But the method which we shall now give for calculating by the aid of Trigonometry the resultant of any number of forces acting at a point assumes that the forces are all in one plane.

56. *Forces act on a particle in one plane: required the magnitude and the direction of their resultant.*

Let $P$, $Q$, $R$,... denote the forces; let $\alpha$, $\beta$, $\gamma$,... denote the angles which their directions respectively make with a fixed straight line drawn through the particle.

By Art. 42 the force $P$ can be resolved into $P\cos\alpha$ and $P\sin\alpha$ along the fixed straight line and at right angles to it respectively; similarly $Q$ can be resolved into $Q\cos\beta$ and $Q\sin\beta$; and $R$ can be resolved into $R\cos\gamma$ and $R\sin\gamma$; and so on.

Then, by algebraical addition of the components which act in the same straight line, we obtain

$$P\cos\alpha + Q\cos\beta + R\cos\gamma + \ldots$$

along the fixed straight line, and

$$P\sin\alpha + Q\sin\beta + R\sin\gamma + \ldots$$

at right angles to the fixed straight line.

We shall denote the former sum by $X$, and the latter by $Y$; hence the given forces are equivalent to the two forces $X$ and $Y$ in directions which are at right angles to each other. Let $K$ be their resultant, and $\theta$ the angle

which the direction of $K$ makes with that of $X$. Then, by Art. 29,

$$K^2 = X^2 + Y^2,$$
$$K \cos \theta = X, \quad K \sin \theta = Y.$$

Thus the magnitude and the direction of the resultant are determined.

57. *To find the conditions of equilibrium when any number of forces act on a particle in one plane.*

The forces will be in equilibrium if their resultant vanishes; that is, by the preceding Article, if $K=0$; and this will be the case if $X=0$ and $Y=0$. These conditions then are sufficient for equilibrium; we may express them in words thus:

*A system of forces acting in one plane on a particle will be in equilibrium if the sum of the resolved parts of the forces along two straight lines at right angles to each other vanishes.*

Conversely, if the forces are in equilibrium $K=0$ and then $X=0$ and $Y=0$; and $X$ denotes the sum of the resolved parts of the forces along *any* straight line. Hence if forces acting in one plane on a particle are in equilibrium, the sum of the resolved parts of the forces along any straight line will vanish.

58. We have hitherto spoken of forces acting at *a point;* but we can often investigate the resultant of forces in one plane which act at *various points* by repeated use of the principles of transference and composition.

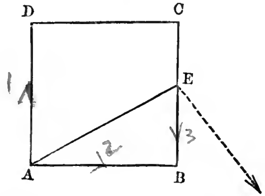
For let there be any number of such forces; take two of them, produce their directions to meet, and suppose the two forces to act at this point; determine their resultant and replace the two forces by this resultant. Then produce the direction of this resultant to meet the direction of one of the remaining forces; and replace the two by their resultant. Proceeding in this way we may represent the resultant of all the forces by carefully drawing the successive diagrams; and we may if we please calculate the numerical value of the resultant.

The only case which could present any difficulty is that in which we should have finally forces in *parallel* directions; and this will be considered in the next Chapter.

We will now give two examples.

(1) Let $ABCD$ be a square; suppose a force of 1 lb. to act along $AD$, a force of 2 lbs. along $AB$, and a force of 3 lbs. along $CB$: required the resultant of the forces.

The forces along $AB$ and $AD$ will have their resultant



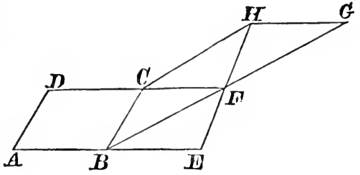

along the straight line $AE$ which is so situated that $EB$ is to $AB$ as 1 lb. is to 2 lbs.; that is, $E$ is the middle point of $BC$. Suppose this resultant applied at $E$; and resolve it again into its components parallel to $AB$ and $AD$. Thus we have a force of 1 lb. along $EC$, and a force of 2 lbs. acting at $E$ parallel to $AB$. There is also a force of 3 lbs. along $CB$. Thus on the whole we have a force of 2 lbs. along $EB$, and also a force of 2 lbs. at $E$ parallel to $AB$.

Hence the resultant of all the forces is $\sqrt{(4+4)}$ lbs., that is $2\sqrt{2}$ lbs.; and its direction passes through $E$, and makes an angle of $45^0$ with $EB$.

(2) $ABCD$ is a parallelogram; forces represented in magnitude and situation by $AB$, $BC$, and $CD$ act on a body: required the resultant of the forces.

The forces $AB$ and $BC$ may be supposed to act at $B$. Produce $AB$ to $E$, so that $BE=AB$. Then the forces $AB$ and $BC$ may also be represented by $BE$ and $BC$; and their resultant will be represented by $BF$, the diagonal of the parallelogram $BEFC$.

The force represented by $CD$ may also be represented by $FC$ which is equal to $CD$. Thus the three given forces




are reduced to the two $BF$ and $FC$, which may be supposed to act at $F$.

Produce $BF$ to $G$, so that $FG=BF$. Then we require the resultant of forces represented by $FG$ and $FC$.

Produce $EF$ to $H$, so that $FH=EF$; join $HG$ and $CH$. Then, by Geometry, $CFGH$ is a parallelogram; and $FH$ represents the resultant of $FG$ and $FC$.

Thus finally the resultant of the forces $AB$, $BC$, and $CD$ is represented in magnitude and in situation by $FH$.

This example deserves the attention of a beginner. We shewed that $BF$ is the resultant of $AB$ and $BC$. Beginners are very apt to say that $BD$ is the resultant of $AB$ and $BC$; this is wrong: $BD$ is the resultant of $BA$ and $BC$, but not of $AB$ and $BC$. Or beginners sometimes say that $AC$ is the resultant of $AB$ and $BC$; this is wrong. $AC$ is the resultant of $AB$ and $AD$, but not of $AB$ and $BC$. Again, beginners sometimes say that the forces $AB$ and $CD$ being equal and opposite balance each other; this is wrong: the forces would balance if they acted in the same straight line, but they do not so act.

We may observe that $BFHC$ is a parallelogram; and that $FH$ is equal to $BC$.

## Examples. III.

1. If three forces represented by the numbers 1, 2, 3 acting in one plane keep a particle at rest, shew that they must all act in the same straight line.

2. Three forces represented by the numbers 1, 2, 3 act on a particle in directions parallel to the sides of an equilateral triangle taken in order: determine their resultant.

3. Can a particle be kept at rest by three forces whose magnitudes are as the numbers 3, 4, and 7?

4. Three forces act at a point parallel to the sides of a triangle, taken in order, and are inversely as the perpendiculars from the angular points of the triangle, on the sides parallel to which the forces act respectively: shew that the forces are in equilibrium.

5. A weight of 25 lbs. hangs at rest, attached to the ends of two strings, the lengths of which are 3 and 4 feet, and the other ends of the strings are fastened at two points in a horizontal line distant 5 feet from each other: find the tension of each string.

6. Three forces acting at a point are in equilibrium; the greatest force is 5 lbs., and the least force is 3 lbs., and the angle between two of the forces is a right angle: find the other force.

7. Two equal forces act at a certain angle on a particle, and have a certain resultant; also if the direction of one of the forces be reversed, and its magnitude be doubled, the resultant is of the same magnitude as before: shew that the resultant of these two resultants is equal to each of them.

8. Two equal forces act at a certain angle on a particle, and have a certain resultant; also if the direction of one of the forces be reversed, and its magnitude be doubled, the resultant is of the same magnitude as before: shew that the two equal forces are inclined at an angle of $60^0$.

9. $AOB$ and $COD$ are chords of a circle which intersect at right angles at $O$; forces are represented in magnitude and direction by $OA$, $OB$, $OC$, $OD$: shew that their resultant is represented in direction by the straight line which joins $O$ to the centre of the circle, and in magnitude by twice this straight line.

10. Eight points are taken on the circumference of a circle at equal distances, and from one of the points straight lines are drawn to the rest: if these straight lines represent forces acting at the point, shew that the direction of the resultant coincides with the diameter through the point, and that its magnitude is four times that diameter.

11. The circumference of a circle is divided into a given even number of equal parts, and from one of the points of division straight lines are drawn to the rest: shew that the direction of the resultant coincides with the diameter through the point.

12. Forces of 3, 4, 5, 6 lbs. respectively act along the straight lines drawn from the centre of a square to the angular points taken in order: find their resultant.

13. Perpendiculars are drawn from any point on the four sides of a rectangle: find the magnitude and the direction of the resultant of the forces represented by the perpendiculars.

14. The circumference of a circle is divided into any number of equal parts; forces are represented in magnitude and direction by straight lines drawn from the centre to the points of division: shew that these forces are in equilibrium.

15. Shew that the result in Example 11 is true also when the number of equal parts is *odd*. Find also the magnitude of the resultant.

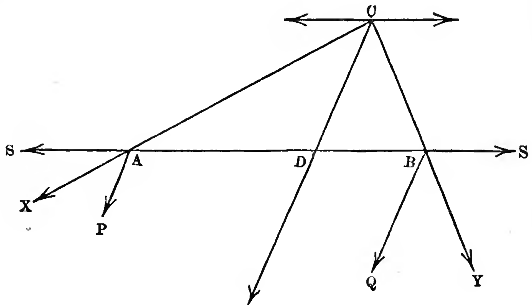
16. $ABCD$ is a parallelogram; forces represented in magnitude and situation by $AB$, $BC$, and $DC$ act on a body: required the resultant of the forces.

## IV. *Resultant of two Parallel Forces.*

59. Forces which have their lines of action parallel are called *parallel forces;* if they tend in the *same* way they may be called *like*, and if they tend in *opposite* ways they may be called *unlike*.

60. *To find the magnitude and the direction of the resultant of two like parallel forces acting on a rigid body.*

Let $P$ and $Q$ be the forces acting at $A$ and $B$ respectively.




The effect of the forces will not be altered if we apply two forces equal in magnitude, and acting in opposite directions along the straight line $AB$. Let $S$ denote each of these forces, and suppose one to act at $A$ along $BA$ produced, and the other at $B$ along $AB$ produced.

Then $P$ and $S$ acting at $A$ are equivalent to a single force $X$ acting in a direction between those of $S$ and $P$; and $Q$ and $S$ acting at $B$ are equivalent to a single force $Y$ acting in a direction between those of $S$ and $Q$.

Produce the directions of $X$ and $Y$ to meet; let them meet at $C$, and draw $CD$ parallel to the directions of $P$ and $Q$, meeting $AB$ at $D$.

Transfer $X$ and $Y$ to $C$, and resolve them along $CD$ and a straight line through $C$ parallel to $AB$; each of the latter components will be equal to $S$, and they will act in opposite directions, and balance each other: the sum of the former components is $P+Q$.

Hence the resultant of the like parallel forces $P$ and $Q$ is $P+Q$, and it acts parallel to the directions of $P$ and $Q$ in a straight line which cuts $AB$ at $D$; so that it may be supposed to act at $D$.

We shall now shew how to determine the point $D$. The sides of the triangle $ADC$ are parallel to the directions of the forces $S$, $P$, $X$; hence, by Art. 36,

$$\frac{AD}{DC}=\frac{S}{P}.$$

Similarly $$\frac{DB}{DC}=\frac{S}{Q}.$$

Therefore $$\frac{AD}{DB}=\frac{Q}{P}.$$

Thus the point $D$ divides $AB$ into segments which are inversely as the forces at $A$ and $B$ respectively.

Let $AB=a$, and $AD=x$; then

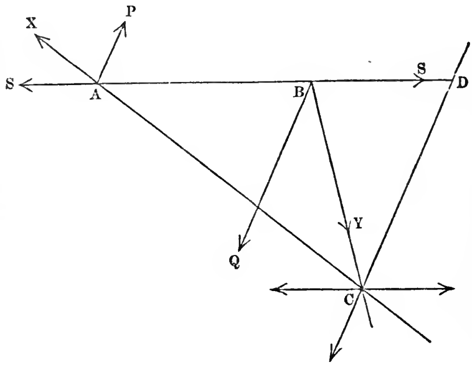
$$\frac{x}{a-x}=\frac{Q}{P};$$

therefore $$Px=Q(a-x),$$

therefore $$x=\frac{Qa}{P+Q}.$$

61. *To find the magnitude and the direction of the resultant of two unlike parallel forces acting on a rigid body.*

Suppose $Q$ the greater of the two forces. By the same method as in the preceding Article we shall arrive at the following conclusion:




The resultant of the unlike parallel forces $P$ and $Q$ is $Q-P$, and it acts parallel to the directions of $P$ and $Q$ in a straight line which cuts $AB$ produced at a point $D$ such that

$$\frac{AD}{BD}=\frac{Q}{P}.$$

Thus $D$ divides $AB$ produced through $B$ into segments which are inversely as the forces at $A$ and $B$ respectively.

Let $AB=a$, and $AD=x$; then

$$\frac{x}{x-a}=\frac{Q}{P};$$

therefore $$Px = Q(x - a);$$

therefore $$x = \frac{Qa}{Q - P}.$$

It will be observed that the results of this Article may be deduced from those of the preceding Article by changing $P$ into $-P$.

62. If three parallel forces keep a rigid body in equilibrium one must be equal and opposite to the resultant of the other two. Hence they must all act in one plane; one of them must be unlike the other two; and its line of action must lie between theirs, dividing the distance between them in the inverse ratio of the two forces.

63. We may find the resultant of any number of parallel forces by repeated application of the process of Arts. 60 and 61. First find the resultant of two of the forces; then find the resultant of this and the third force; and so on.

64. There is one case in which we are unable to find a single resultant for two parallel forces, namely, when the forces are *equal* and *unlike*. The process of Art. 61 will not apply to this case; for the lines of action of the forces $X$ and $Y$ are then parallel, so that the points $C$ and $D$ do not exist.

Two such forces are usually called a *couple*. The theory of couples includes some important propositions; it will be sufficient for our purpose to demonstrate one of these: some preliminary definitions and remarks will be required.

65. A *couple* consists of two parallel forces which are *equal* and *unlike*.

The *arm* of a couple is the perpendicular distance between the lines of action of its forces.

The *moment* of a couple is the product of one of the equal forces into the arm; that is, the *number* which ex-

presses the force must be multiplied by the *number* which expresses the arm to produce the moment.

66. The student will readily conceive that the tendency of a couple which acts on a free rigid body is to make the body turn round; and it is shewn in works on the higher parts of mechanics that such is the case: the rotation begins to take place round a straight line which always passes through a certain point in the body called its *centre of gravity*, but is not necessarily perpendicular to the plane of the couple.

67. Two couples in the same plane may differ as to the direction in which they tend to turn the body round on which they act.

Take for example the couple denoted in the figure of the next Article by the two equal forces $P$, and that denoted by the two equal forces $Q$. Suppose a board $ABCD$ capable of turning in its own plane round a pivot fixed at any point within $ABCD$. The couple $Q$, $Q$ tends to turn the board in the same way as the hands of a watch revolve, and the couple $P$, $P$ tends to turn the board in the opposite way.

Couples which tend to turn a body round in the *same* way may be called *like*, and couples which tend to turn a body round in *opposite* ways may be called *unlike*.

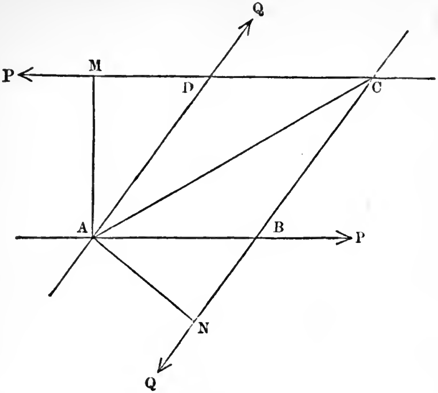
68. *Two unlike couples in the same plane will balance each other if their moments are equal.*

Let each force of one couple be $P$, and each force of the other couple $Q$. Let $ABCD$ be the parallelogram formed by the lines of action of the forces.

Draw $AM$ perpendicular to $CD$, and $AN$ perpendicular to $CB$.

By hypothesis the moments of the couples are equal, that is,

$$P \times AM = Q \times AN;$$




therefore $$\frac{P}{Q} = \frac{AN}{AM}$$

$$= \frac{AB}{AD}, \text{ by Euclid VI. 4.}$$

Hence, by the parallelogram of forces, the resultant of $P$ and $Q$ at $A$ acts in the direction $AC$.

Similarly, the resultant of $P$ and $Q$ at $C$ acts in the direction $CA$, and is equal to the former resultant in magnitude; so that the two resultants balance each other.

Hence the two couples balance each other.

Since two unlike couples of equal moment in the same plane balance each other, it follows that *two like couples of equal moment in the same plane produce equal effects.*

69. The demonstration of the preceding Article assumes that the forces of one couple are not parallel to those of the other. When the four forces are all parallel, the theorem may be shewn to be true by the aid of Arts. 60 and 61. Or we may proceed thus. Suppose the couples unlike, and all the forces parallel; take a couple in the

same plane, and of equal moment, with its forces not parallel to the four forces. Then by Art. 68 this new couple balances one of the two couples and is equivalent to the other. Therefore the two couples balance each other.

70. Hence we see that two like couples of equal moments in the same plane are equivalent to a single like couple in that plane of double moment. And any number of like couples in the same plane are equivalent to a like couple in that plane with a moment equal to the sum of the moments of these couples.

Hence finally, if any number of couples act in one plane, some of one kind and some of the other, they are equivalent to a single couple in that plane, having a moment equal to the difference of the sums of the moments of the two kinds, and being of the same kind as the couples which have the greater sum of moments.

71. *A single force and a couple in one plane are equivalent to the same single force acting in a direction parallel to its original direction.*

Let $P$ denote the single force, $Q$ each force of the couple.

If the directions of all the forces are parallel, $P$ combined with the *like* force of the couple will produce a resultant $P+Q$ also parallel to the former force. Then $P+Q$ combined with the other force of the couple will produce a resultant $P$ parallel and like to the original $P$.

If the directions of all the forces are not parallel, let $A$ denote the point at which the line of action of $P$ crosses that of one force $Q$ of the couple. Form the resultant of $P$ and this $Q$, and denote the resultant by $R$. Let $B$ denote the point at which the line of action of $R$ crosses that of the other force $Q$ of the couple; and suppose $R$ to act at this point. Resolve $R$ into its components $P$ and $Q$. The two forces $Q$ balance each other, leaving the force $P$ acting at $B$ parallel to its original direction.

## EXAMPLES. IV.

1. $ABCD$ is a square. A force of 3 lbs. acts from $A$ to $B$, a force of 4 lbs. from $B$ to $C$, a force of 6 lbs. from $D$ to $C$, and a force of 5 lbs. from $A$ to $D$: find the magnitude and the direction of the resultant force.

2. Two men carry a weight of 152 lbs. between them on a pole, resting on one shoulder of each; the weight is three times as far from one man as from the other: find how much weight each supports, the weight of the pole being disregarded.

3. A man supports two weights slung on the ends of a stick 40 inches long placed across his shoulder: if one weight be two-thirds of the other, find the point of support, the weight of the stick being disregarded.

4. $ABC$ is a triangle, and $O$ any point within it; like parallel forces $P$ and $Q$ act at $A$ and $B$ such that $P$ is to $Q$ as the area of $BOC$ is to the area of $AOC$: shew that the resultant acts at the point where $CO$ produced meets $AB$.

5. If the point $O$ be *outside* the triangle and the forces $P$ and $Q$ in the same proportion as in Example 4, shew that the result is still true, provided $P$ and $Q$ be like or unlike according as the intersection of $CO$ and $AB$ is on $AB$ or on $AB$ produced.

6. $ABC$ is a triangle, and $O$ any point within it; like parallel forces act at $A$, $B$, and $C$ which are proportional to the areas $BOC$, $COA$, and $AOB$ respectively: shew that the resultant acts at $O$.

7. If the point $O$ be *outside* the triangle, and the forces in the same proportion as in Example 6, but a certain one of the three be unlike the other two, shew that the resultant acts at $O$.

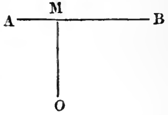
8. $P$ and $Q$ are like parallel forces; an unlike parallel force $P+Q$ acts in the same plane at perpendicular distances $a$ and $b$ respectively from the two former, and between them: determine the moment of the couple which results from the three forces.

9. Like parallel forces, each equal to $P$, act at three of the corners of a square perpendicular to the square; at the other corner such a force acts that the whole system is a couple: determine the moment of the couple.

## V. *Moments.*

72. The product of a force into the perpendicular drawn on its line of action from any point is called the *moment* of the force with respect to that point.

Thus suppose $AB$ the line of action of a force, $O$ any point, and $OM$ the perpendicular from $O$ on $AB$. Then if $P$ denote the force the moment with respect to $O$ is $P \times OM$.




Hence the moment of a force never vanishes except when the point about which the moment is taken is on the line of action of the force.

73. Suppose that $OM$ were a rod which could turn round $O$ in the plane of the paper; if a force were to act at $M$ in the direction $AB$ it would tend to make the rod turn round $O$ in the same direction as the hands of a watch revolve; if the force were to act in the direction $BA$ it would tend to make the rod turn round $O$ in the opposite direction. It is found very convenient to distinguish between these two kinds of moment; and thus moments of one kind are called *positive*, and moments of the other kind *negative*. It is indifferent in any investigation which kind of moment we consider positive, and which negative; but when we have made a choice we must keep to it during that investigation.

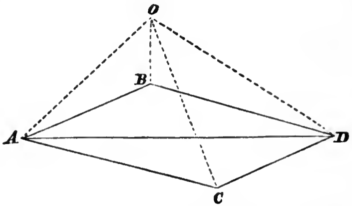
74. Since the area of any triangle is equal to half the product of the base into the altitude, the moment of a force may be geometrically represented by twice the area of a triangle having for its base the straight line which represents the force, and for its vertex the point about which the moments are taken.

We shall now give a very important proposition respecting moments.

75. *The algebraical sum of the moments of two forces round any point in the plane containing the two forces is equal to the moment of their resultant.*

Let $AB$, $AC$ represent two forces; complete the parallelogram $ABCD$ and draw the diagonal $AD$, which will represent the resultant force. Let $O$ be the point round which the moments are to be taken; join $OA$, $OB$, $OC$, $OD$.

I. Let $O$ fall *without* the angle $BAC$ and that which is vertically opposite to it.



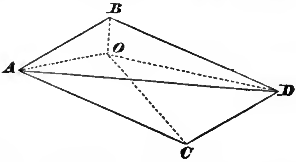

The area of the triangle $AOC$ is equal to half the product of the base $AC$ into the perpendicular on it from $O$; and this perpendicular is equal to the sum of the perpendiculars from $A$ and from $O$ on $BD$. Now $BD$ is equal to $AC$; so that the product of $AC$ into the perpendicular on it from $O$ is equal to the product of $BD$ into the sum of the perpendiculars on it from $A$ and from $O$.

Thus triangle $AOC$ = triangle $ABD$ + triangle $OBD$
= triangle $AOD$ − triangle $AOB$;
therefore triangle $AOC$ + triangle $AOB$ = triangle $AOD$.

Hence moment of $AC$ + moment of $AB$ = moment of $AD$.

II. Let $O$ fall *within* the angle $BAC$ or that which is vertically opposite to it.

The area of the triangle $AOC$ is equal to half the product of the base $AC$ into the perpendicular on it from $O$; and this perpendicular is equal to the difference of the perpendiculars from $A$ and from $O$ on $BD$. Now $BD$ is equal to $AC$; so that the product of $AC$ into the perpendicular on it from $O$ is equal to the product of $BD$ into the difference of the perpendiculars on it from $A$ and from $O$.




Thus triangle $AOC$ = triangle $ABD$ − triangle $OBD$
= triangle $AOB$ + triangle $AOD$;
therefore triangle $AOC$ − triangle $AOB$ = triangle $AOD$.

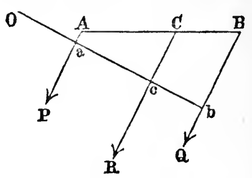
Hence moment of $AC$ − moment of $AB$ = moment of $AD$.

The moments of $AC$ and $AB$ round $O$ are now of *opposite* kinds: thus the moment of the resultant is always

equal to the algebraical sum of the moments of the components.

76. The preceding Article applies to the case in which the lines of action of the forces *meet:* we have still to shew that the proposition holds for *parallel forces.*

Let $P$ and $Q$ be like parallel forces, acting at $A$ and $B$; let $R$ be their resultant, $C$ the point where the direction of the resultant cuts $AB$. Take any point $O$, not between the lines of action of the forces, and draw a perpendicular from $O$ on the lines of action of $P$, $Q$, and $R$, meeting them at $a$, $b$, and $c$ respectively.




Now $\frac{P}{Q}=\frac{CB}{CA}$, by Art. 60, $=\frac{cb}{ca}$, by Euclid VI. 2;

therefore $$P \times ca = Q \times cb.$$

And $P \times Oa + Q \times Ob = P(Oc - ca) + Q(Oc + cb)$
$= (P+Q)\, Oc - P \times ca + Q \times cb = (P+Q)\, Oc = R \times Oc.$

Thus the required result is obtained in this case.

Similarly, it may be shewn that when $O$ falls between the lines of action of $P$ and $Q$ the moment of $R$ is equal to the *arithmetical* difference of the moments of $P$ and $Q$.

Thus for two *like* parallel forces the moment of the resultant is always equal to the algebraical sum of the moments of the components.

In a similar manner the proposition may be established for the case of *unlike* parallel forces.

77. *The algebraical sum of the moments of the two forces which form a couple is constant round any point in the plane of the couple.*

If the point is *between* the lines of action of the forces the moments of the two forces are of the *same* kind, and their sum is equal to the product of one of the forces into the perpendicular distance between the lines of action.

If the point is *not* between the lines of action of the forces the moments of the two forces are of *opposite* kinds,

and their arithmetical difference is equal to the product of one of the forces into the perpendicular distance between the lines of action.

Thus in every case the algebraical sum of the moments of the forces of a couple round a point in the plane of the couple is equal to the product of one force into the perpendicular distance between the lines of action; that is, to the moment of the couple. See Art. 65.

78. When two forces act in one plane one of three cases must hold. Either the forces balance each other so that their resultant is zero, or they have a single resultant, or they form a couple. The algebraical sum of the moments of the forces about a point in the plane always vanishes in the first case, vanishes in the second case only when the point is on the line of action of the resultant, and never vanishes in the third case.

79. The result of Art. 75 can be extended to the case of any number of forces acting in one plane at a point. Take two of the forces; the algebraical sum of their moments round any point is equal to the moment of their resultant. Replace the two forces by their resultant; then apply Art. 75 again to this resultant and the third force. And so on.

80. Similarly, by the aid of Art. 63 we may extend Art. 76 to the case of any number of parallel forces acting in one plane.

81. If any number of forces in one plane act on a particle either they are in equilibrium or they have a single resultant. The algebraical sum of the moments round a point in the plane always vanishes in the former case, and in the latter case vanishes only when the point is on the line of action of the resultant.

82. Hence we may use the following instead of Art. 57: *Forces acting in one plane on a particle will be in equilibrium if the algebraical sum of the moments vanishes round any two points in the plane which are not situated on a straight line passing through the particle.*

Conversely, if the forces are in equilibrium the algebraical sum of the moments will vanish round any point in the plane.

## EXAMPLES. V.

1. $P$ and $Q$ are fixed points on the circumference of a circle; $QA$ and $QB$ are any two chords at right angles to each other, on opposite sides of $QP$: if $QA$ and $QB$ denote forces, shew that the difference of their moments with respect to $P$ is constant.

2. If two or more forces act in one plane on a particle, shew that the algebraical sum of their moments round a point in the plane remains unchanged when the point moves parallel to a certain straight line.

3. If the algebraical sum of the moments of forces acting in one plane on a particle has the same value for two points in the plane, then the straight line which joins these two points is parallel to the resultant force.

4. $ABC$ is a triangle; $D$, $E$, $F$ are the middle points of the sides opposite to $A$, $B$, $C$ respectively: shew by taking moments round $A$, $B$, and $C$, that forces denoted by $AD$, $BE$, and $CF$ are in equilibrium.

5. Forces are denoted by the perpendiculars drawn from the angular points of a triangle on the opposite sides: shew by taking moments round the angular points that the forces will not be in equilibrium unless the triangle is equilateral.

6. Forces act at the angular points of a triangle along the perpendiculars drawn from the angular points on the respectively opposite sides, each force being proportional to the side to which it is perpendicular: shew by taking moments round the angular points that the forces are in equilibrium.

7. $ABC$ is a triangle, $O$ any point within it; $AO$, $BO$, and $CO$ produced cut the respectively opposite sides at $H$, $I$, and $K$: shew that forces denoted by $AH$, $BI$, and $CK$, cannot be in equilibrium unless $H$, $I$, and $K$ are the middle points of the respective sides.

8. $ABC$ is a triangle, $O$ any point within it; straight lines are drawn through $O$ parallel to the sides and terminated by the sides: shew that forces denoted by these straight lines cannot be in equilibrium unless each straight line is bisected at $O$.

## VI. *Forces in one Plane.*

83. In the present chapter we shall investigate the conditions of equilibrium of a system of forces acting in one plane on a rigid body.

84. *A system of forces acting in one plane on a rigid body, if not in equilibrium, must be equivalent to a single resultant or to a couple.*

For take any two of the forces of the system, and determine their resultant; then combine this resultant with another force of the system; and so on. By proceeding in this way we must obtain finally a single resultant or a couple; unless one force of the system is equal and opposite to the resultant of all the others, so that the whole system is in equilibrium.

85. *If a system of forces acting in one plane on a rigid body is equivalent to a single resultant, the moment of the resultant round any point in the plane is equal to the algebraical sum of the moments of the forces.*

This proposition is established by repeated applications of Arts. 75 and 76. Take any two of the forces; then the moment of their resultant round any point in the plane is equal to the algebraical sum of the moments of the two forces. Combine the resultant of these two forces with another of the forces; then the moment of their resultant is equal to the algebraical sum of the moments of the components, that is, to the algebraical sum of the moments of the three forces of the system. And so on.

Hence the algebraical sum of the moments of the forces round any point in the plane will not vanish unless the point about which the moments are taken is on the line of action of the resultant.

86. *If a system of forces acting in one plane on a rigid body is in equilibrium, the algebraical sum of the moments round any point in the plane vanishes.*

For when the system of forces is in equilibrium, one of the forces is equal and opposite to the resultant of all the

others. Hence the moment of the one force round any point in the plane is equal and contrary to the moment of the resultant of all the other forces; see Art. 78. Thus, by Art. 85, the moment of the one force is equal and contrary to the algebraical sum of the moments of all the other forces. Therefore the algebraical sum of the moments of all the forces vanishes.

87. *If a system of forces acting in one plane on a rigid body is equivalent to a couple, the algebraical sum of the moments of the forces round any point in the plane is equal to some constant which is not zero.*

For proceeding as in Art. 85 we have the algebraical sum of the moments of the forces equal to the moment of the couple; and the moment of the couple is constant for all points of the plane, namely, equal to the product of one force of the couple into the perpendicular distance between the two forces.

*Conversely, if the algebraical sum of the moments of the forces round three points in the plane not in the same straight line is constant, and not zero, the system of forces is equivalent to a couple.*

For if the system is not equivalent to a couple it must be in equilibrium or reduce to a single force; by Art. 84. If it were in equilibrium the sum of the moments would be zero; by Art. 86. If it reduced to a single force the sum of the moments could not have a constant value except for points in a straight line parallel to the direction of the single force; by Art. 85.

88. *A system of forces acting in one plane on a rigid body will be in equilibrium if the algebraical sum of the moments of the forces vanishes round three points in the plane which are not in a straight line.*

For if the system of forces be not in equilibrium, it is equivalent to a single resultant or to a couple.

In the present case the system of forces cannot be equivalent to a couple; for then the algebraical sum of the moments would not vanish for any point in the plane.

Nor can the system of forces be equivalent to a single resultant; for then the algebraical sum of the moments

would vanish only for points on the line of action of the resultant.

Hence the system of forces must be in equilibrium.

89. The preceding Article gives the *conditions of equilibrium* of a system of forces acting in one plane on a rigid body; if these conditions are satisfied the body is in equilibrium, or in other words these conditions are *sufficient* for equilibrium. And these conditions are *necessary* for equilibrium; because we have shewn in Art. 86 that they must hold if there be equilibrium.

We shall give a proposition in the next Article which will enable us to put the conditions of equilibrium in another form.

90. *If a system of forces acting in one plane on a rigid body is in equilibrium the algebraical sum of the forces resolved parallel to any fixed straight line vanishes.*

For when the system of forces is in equilibrium, one of the forces is equal and opposite to the resultant of all the others. Hence the resolved part of the one force parallel to any fixed straight line is equal and opposite to the algebraical sum of the resolved parts parallel to the fixed straight line of all the other forces; see Arts. 44 and 52. Therefore the algebraical sum of the forces resolved parallel to any fixed straight line vanishes.

91. *A system of forces acting in one plane on a rigid body will be in equilibrium if the algebraical sum of the moments round two points in the plane vanishes, and the algebraical sum of the forces resolved parallel to the straight line which joins these points also vanishes.*

For if the system of forces be not in equilibrium it is equivalent to a single resultant or to a couple.

In the present case the system of forces cannot be equivalent to a couple; for then the algebraical sum of the moments would not vanish for any point in the plane.

Nor can the system of forces be equivalent to a single resultant; for then the algebraical sum of the moments would vanish only for points on the line of action of the

resultant, so that the resultant would act along the straight line joining the two points: but the algebraical sum of the forces resolved parallel to this straight line vanishes by supposition; therefore there cannot be a resultant acting along this straight line: see Arts. 44 and 52.

Hence the system of forces must be in equilibrium.

92. In the proposition of the preceding Article we speak of forces resolved parallel to the straight line which joins two points; this means of course that every force is resolved into two components, one parallel to this straight line, and the other parallel to another fixed straight line: the second straight line is not *necessarily* at right angles to the first, although we may for convenience generally take it so.

We shew in the preceding Article that the conditions of equilibrium there stated are *sufficient;* and we see by Arts. 86 and 90 that they are *necessary.*

There is still another form in which the conditions of equilibrium of a system of forces acting in one plane on a rigid body may be put: and this we shall now give.

93. *A system of forces acting in one plane on a rigid body will be in equilibrium if the algebraical sum of the forces resolved parallel to two fixed straight lines in the plane vanishes, and the algebraical sum of the moments round any point in the plane also vanishes.*

For if the system of forces be not in equilibrium it is equivalent to a single resultant or to a couple.

In the present case the system of forces cannot be equivalent to a couple; for then the algebraical sum of the moments would not vanish round any point in the plane.

Nor can the system of forces be equivalent to a single resultant; for, by supposition, the algebraical sum of the forces resolved parallel to two fixed straight lines in the plane vanishes, and therefore the resolved part of the single resultant parallel to these two straight lines would vanish: so that the resultant would vanish. See Arts. 44 and 52.

Hence the system of forces must be in equilibrium.

94. In the proposition of the preceding Article the two fixed straight lines are not *necessarily* at right angles, although it may generally be convenient to take them so.

We shew in the preceding Article that the conditions of equilibrium there stated are *sufficient;* and we see by Arts. 86 and 90 that they are *necessary.*

We have thus presented in three forms the conditions of equilibrium of a system of forces acting in one plane on a rigid body; see Arts. 88, 91, and 93: the last form is generally the most convenient to use.

95. The preceding Articles relate to any system of forces acting in one plane on a rigid body; the particular case in which the forces are *parallel* deserves separate notice; for some of the general propositions may then be simplified. We have the following results:

*A system of parallel forces acting in one plane on a rigid body, if not in equilibrium, must be equivalent to a single resultant parallel to the forces or to a couple.* See Arts. 63 and 84.

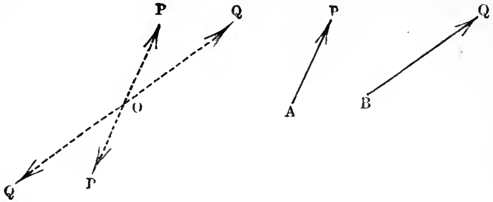
*A system of parallel forces acting in one plane on a rigid body will be in equilibrium if the algebraical sum of the moments of the forces vanishes round two points in the plane, which are not in a straight line parallel to the direction of the forces.* See Art. 88.

*A system of parallel forces acting in one plane on a rigid body will be in equilibrium if the algebraical sum of the forces vanishes, and also the algebraical sum of the moments of the forces round any point in the plane.* See Art. 91 or 93.

96. Many of the results of the present Chapter depend on the theorem of Art. 84, and although the simple reasoning of that Article is quite satisfactory, it may be desirable to give another investigation. Accordingly we shall now demonstrate a new theorem which includes that of Art. 84.

97. *A system of forces acting in one plane on a rigid body can in general be reduced to a couple and a single force acting at an arbitrary point in the plane.*

Let $P$ acting at $A$ be one of the forces of the system.




At any arbitrary point $O$ in the plane apply two forces each equal and parallel to $P$, in opposite directions. This will not alter the action of the system of forces. Thus, instead of $P$ at $A$ we have $P$ at $O$ and a couple formed by $P$ at $A$ and $P$ at $O$.

Let $Q$ acting at $B$ be another of the forces of the system. Then, as before, we may replace $Q$ at $B$ by $Q$ at $O$ and a couple formed by $Q$ at $B$ and $Q$ at $O$.

Proceeding in this way we can replace the given system of forces by the following:

(1) A system of forces at $O$ which are respectively parallel and equal to the original forces; this system may be combined into a single force at $O$.

(2) A set of couples in the plane which may be combined into a single couple by Art. 70. As the moment round $O$ of the force at $P$ is equal to the moment of the couple consisting of $P$ at $A$ and $P$ at $O$, it follows that the moment of the single couple thus obtained is equal to the sum of the moments of the forces.

Thus in general a system of forces acting in one plane on a rigid body may be reduced to a couple and a single force; the latter acting through any arbitrary point in the plane, and being equal in magnitude and parallel in direction to what would be the resultant of all the forces if they acted at one point parallel respectively to their original directions.

It may happen that the couple vanishes and then the system can be reduced to a single force; or the single force may vanish and then the system reduces to a couple; or both the single force and the couple may vanish and then the system is in equilibrium.

If neither the couple nor the single force vanishes they can be reduced to a single force by Art. 71.

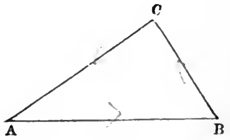
98. We will now give some examples which illustrate the subject of the present Chapter.

(1) *If a system of forces is represented in magnitude and position by the sides of a plane polygon taken in order the system is equivalent to a couple.*

The sum of the moments of the forces is constant round any point in the plane. For if the point be taken *within* the polygon the moments are all of the same kind, and their sum is represented by twice the area of the polygon. And if the point be taken *without* the polygon the moments are not all of the same kind, but their algebraical sum is constant, being still represented by twice the area of the polygon.

Since the sum of the moments is equal to a constant which is not zero, the system of forces is equivalent to a couple; see Art. 87.

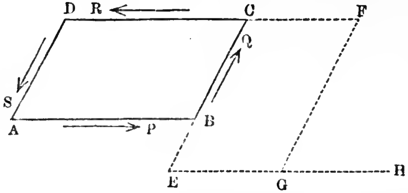
(2) ABC *is a triangle; a force* P *acts from* A *to* B, *a force* Q *from* B *to* C, *and a force* R *from* A *to* C: *required the relation between the forces in order that they may reduce to a single resultant passing through the centre of the circle inscribed in the triangle.*




The algebraical sum of the moments round the centre of the inscribed circle must vanish; see Art. 85.

Let $r$ denote the radius of the inscribed circle, then $Pr + Qr = Rr$; therefore $R = P + Q$. This is the necessary and sufficient condition.

(3) Forces $P$, $Q$, $R$, $S$ act in order round the sides of a parallelogram: required the direction of the resultant.




The resultant of $P$ and $R$ will be a force equal to $P-R$ acting parallel to $AB$, through the point $E$ on $CB$ produced through $B$, such that

$$\frac{BE}{CE}=\frac{R}{P}.$$

The resultant of $Q$ and $S$ will be a force equal to $Q-S$ acting parallel to $BC$, through the point $F$ on $DC$ produced through $C$, such that

$$\frac{CF}{DF}=\frac{S}{Q}.$$

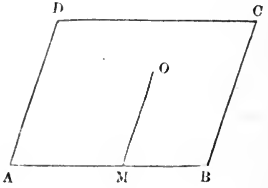
Draw $EG$ parallel to $AB$, and $FG$ parallel to $CB$. Then the resultant of all the four forces passes through $G$. Produce $EG$ to $H$ so that $GH$ may be to $GF$ as $P-R$ is to $Q-S$. Then the straight line joining $G$ to the middle point of $FH$ is the direction of the resultant.

We have supposed $P$ greater than $R$ and $Q$ greater than $S$; it is easy to make the necessary modifications for any other case.

Or we may proceed thus: Let $AB=h$, $AD=k$; from any point $O$ draw $OM$ parallel to $CB$; let

$$AM=x, \quad OM=y.$$

We can now express the moments of the forces round $O$.



The moment of $P = P \times OM \sin OMB = Py \sin A$,

the moment of $Q = Q \times MB \sin A = Q(h - x) \sin A$,

the moment of $R = R \times (BC - OM) \sin A = R(k - y) \sin A$,

the moment of $S = S \times AM \sin A = Sx \sin A$.

Now if $O$ be a point on the line of action of the resultant the algebraical sum of the moments of the forces round $O$ vanishes, that is,

$$Py + Q(h - x) + R(k - y) + Sx = 0,$$

or

$$(P - R)y - (Q - S)x + Qh + Rk = 0.$$

This gives the relation which must hold between $AM$ and $MO$, when $O$ is a point on the line of action of the resultant.

For example, suppose $O$ to be on the straight line $AD$; then $x = 0$: thus we get

$$y = \frac{Qh + Rk}{R - P},$$

which determines the point where the direction of the resultant cuts $AD$. Similarly we may determine the point where this direction cuts $AB$.

## EXAMPLES. VI.

1. $ABCD$ is a square. A force of 3 lbs. acts from $A$ to $B$, a force of 4 lbs. from $B$ to $C$, and a force of 5 lbs. from $C$ to $D$: find the single force which will preserve equilibrium.

2. A man carries a bundle at the end of a stick over his shoulder: as the portion of the stick between his shoulder and his hand is shortened, shew that the pressure on his shoulder is increased. Does this change alter his pressure on the ground?

3. If forces in one plane reduce to a couple, shew that if they were made to act on a particle, retaining their mutual inclinations, they would keep the particle at rest.

4. $ABC$ is a triangle; $H$, $I$, and $K$ are points in the sides $BC$, $CA$, and $AB$ respectively, such that

$$\frac{BH}{HC} = \frac{CI}{IA} = \frac{AK}{KB}:$$

shew by taking moments round $A$, $B$, $C$ that forces denoted by $AH$, $BI$, and $CK$ are equivalent to a couple, except when $H$, $I$, and $K$ are the middle points of the sides, and then the forces are in equilibrium.

5. $A$ and $B$ are fixed points; at any point $C$, in the arc of a circle described on $AB$ as a chord, two forces act, namely, $P$ along $CA$ and $Q$ along $CB$: shew that their resultant passes through a fixed point on the other arc which makes up the complete circle.

6. $ABCD$ is a quadrilateral inscribed in a circle: if forces $P$, $Q$, $R$ act in directions $AB$, $AD$, $CA$ so that $P : Q : R$ as $CD : BC : BD$, shew that they are in equilibrium.

7. Two forces are denoted by $MA$ and $MB$, and two others by $NC$ and $ND$: shew that the four forces cannot be in equilibrium unless $MN$ bisects both $AB$ and $CD$.

8. Find a point within a quadrilateral such that if forces be represented by straight lines drawn from it to the angular points of the quadrilateral the forces will be in equilibrium.

9. Forces proportional to 1, $\sqrt{3}$, and 2 act at a point and are in equilibrium: find the angles between their lines of action.

10. If two equal forces $P$ and $P$ acting at an angle of $60^0$ have the same resultant as two equal forces $Q$ and $Q$ acting at right angles, shew that $P$ is to $Q$ as $\sqrt{2}$ is to $\sqrt{3}$.

11. $C$ and $B$ are fixed points; $CA$ and $CB$ represent two forces: if $A$ move along any straight line shew that the extremity of the straight line which represents the resultant moves along a parallel straight line.

12. Forces denoted by the sides of a polygon, except one side, act in order: shew that they are equivalent to a single resultant which is parallel to the omitted side.

## VII. *Constrained Body.*

99. A body is said to be *constrained* when the manner in which it can move is restricted. A simple example is that in which a body can only turn round a fixed axis, that is, can receive no other motion. In such cases forces may act on the body besides the restraints which restrict the motion, and we may require to know the conditions which must hold among these forces in order to ensure the equilibrium of the body.

100. *When a body can only turn round a fixed axis and is acted on by a system of forces in a plane perpendicular to the axis, such that the algebraical sum of the moments of the forces round the point where the axis meets the plane vanishes, the body will be in equilibrium.*

If the system of forces be not in equilibrium it is equivalent to a single resultant or a couple.

In the present case the system of forces cannot be equivalent to a couple; for then the algebraical sum of the moments would not vanish for any point in the plane.

Suppose that the system of forces is equivalent to a single resultant. Since the algebraical sum of the moments of the forces vanishes round the point where the axis meets the plane, the line of action of the resultant must pass through the point. Therefore the resultant has no tendency to turn the body round the axis; and the body is therefore in equilibrium.

101. The investigation of the preceding Article shews that the condition there stated is *sufficient* for equilibrium. The condition is also *necessary* for equilibrium; for if the condition does not hold, the system of forces is equivalent either to a couple or to a single resultant which does not pass through the axis, and in either case the body would be set in motion round the axis.

102. The most simple case of the preceding two Articles is that of the *lever*. A lever is a rigid body which is moveable in one plane about a point which is called the *fulcrum*, and is acted on by forces which tend

to turn it round the fulcrum. In order that the lever may be in equilibrium the moments of the two forces round the fulcrum must be equal and contrary, by Art. 101. Hence the condition of equilibrium stated in Art. 100 is often called the *Principle of the Lever.*

103. A body which is not constrained is called a *free* body. From considering the equilibrium of a constrained body we may render our conception of the equilibrium of a free body more distinct. Any condition which is *necessary* for the equilibrium of a constrained body will also be necessary for the equilibrium of a free body; although a condition which may be *sufficient* in the former case will not generally be sufficient in the latter case.

For example, in Art. 86 a certain principle is established with respect to the equilibrium of a free rigid body, and the investigation of Art. 100 shews us the interpretation of the principle. Suppose a body in equilibrium under the action of a system of forces in one plane. Imagine two points in the body, which lie in a straight line perpendicular to the plane, to *become fixed.* This cannot disturb the equilibrium, for we do not communicate any motion to the body by fixing two points in it; we merely *restrict* to some extent its possible motion. The body has still the power of turning round the straight line which joins the fixed points; and, by Art. 101, the body will not be in equilibrium unless the algebraical sum of the moments of the forces round the point where the straight line cuts the plane vanishes.

104. Suppose a body can only turn round a fixed axis, and that it is acted on by forces which are not all in one plane perpendicular to the axis; a strict demonstration of the condition of equilibrium is rather beyond our present range, but by assuming some principles which are nearly self-evident we shall be able to give a sufficient investigation.

First suppose the forces to consist of various systems in planes which are all perpendicular to the axis. It may be assumed as nearly self-evident that the tendency of the systems to set the body in motion will not be altered if all the other planes are made to coincide with one of

them; and then the forces reduce to a system in one plane perpendicular to the axis, and Arts. 100 and 101 apply.

Next suppose the forces to be any whatever. Resolve each force into two components at right angles to each other; *one component being parallel to the fixed axis.* It may be assumed as nearly self-evident that the components parallel to the axis have no tendency to set the body in motion *round the axis;* and they may accordingly be left out of consideration.

The other components form various systems of forces in planes which are perpendicular to the axis; and, as in the first case, they may be supposed all to act in one plane, and Arts. 100 and 101 apply.

105. Suppose a body capable of moving only in such a manner that all points of the body describe parallel straight lines. For example, two fixed rigid parallel straight rods may pass through the body, and so the body be only capable of sliding along the rods. Suppose also that a system of forces acts on the body. Resolve each force into two components at right angles to each other, one component being parallel to the fixed rods. Then the necessary and sufficient condition of equilibrium is that the sum of the components parallel to the fixed rods, that is to the direction of possible motion, should vanish.

If there were only one rigid straight rod passing through it the body could both slide and turn round; in such a case, besides the condition just obtained, that of Art. 104 must also hold for equilibrium.

We see from these cases the interpretation of the condition in Art. 90 relative to the equilibrium of a free rigid body.

106. *When three forces maintain a body in equilibrium their lines of action must lie in the same plane.*

Suppose a body in equilibrium under the action of three forces. Imagine two points in the body, one on the line of action of one force, and the other on the line of action of another force, to become fixed, the points being so taken that the straight line which joins them is not parallel

to the line of action of the third force. This cannot disturb the equilibrium. The body has still the power of turning round the straight line which joins the fixed points, as an axis, and it will not be in equilibrium unless the line of action of the third force pass through the axis.

Thus *any* straight line which meets the lines of action of two of the forces, and is not parallel to the line of action of the third force, must meet that line of action; and therefore all the three lines of action must lie in one plane.

By combining this result with those in Arts. 41 and 62 we have a complete account of the conditions of equilibrium of a rigid body when acted on by *three* forces.

107. In the present work on Mechanics we have begun with the *Parallelogram of Forces* and have deduced the *Principle of the Lever;* this is the course which is now generally adopted. Formerly it was usual to begin with the *Principle of the Lever* and to deduce the *Parallelogram of Forces.* We will briefly indicate the principal steps of the process.

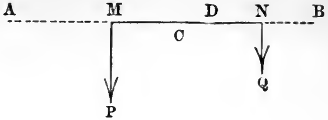
Various axioms are laid down: for example the following: *Equal forces acting at right angles at the extremities of equal arms of a lever, exert equal efforts to turn the lever round.*

From these axioms certain propositions are deduced; for example the following: *A horizontal rod or cylinder of uniform density will produce the same effect by its weight as if it were collected at its middle point.*

In this way the Principle of the Lever is established, first for a straight lever, and then for a lever of any form. We will give one proposition as an example of this method in the next Article, and in the following Article we will shew how the Parallelogram of Forces is deduced.

108. *Two forces acting at right angles on a straight lever on opposite sides of the fulcrum will balance each other if they are inversely proportional to their distances from the fulcrum and tend to turn the lever round in contrary directions.*

Let the forces $P$ and $Q$ which act at $M$ and $N$ at right angles to a straight lever on opposite sides of the fulcrum $C$ be such that




$$\frac{P}{Q}=\frac{CN}{CM},$$

and let them be *like* forces, so that they tend to turn the lever in contrary directions : they will balance each other.

If $P=Q$, the proposition is true by the axiom stated in Art. 107.

If $P$ be not equal to $Q$, suppose $P$ the greater.

On $NM$ take $ND=MC$; then $NC=MD$. Make $MA=MD$, and $NB=ND$.

Let the forces $P$ and $Q$ be measured by the weights which they would support; and let $AB$ be a uniform rod of weight equal to $P+Q$.

Now $$CA=CM+MA=ND+MD=MN;$$
$$CB=CN+NB=MD+ND=MN;$$

therefore $AB$ is bisected at $C$.

And $$\frac{AD}{DB}=\frac{2MD}{2ND}=\frac{2NC}{2MC}=\frac{P}{Q};$$

therefore $$\frac{AD}{AD+DB}=\frac{P}{P+Q}.$$

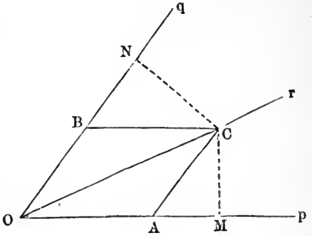
But the weight of $AB$ is $P+Q$; and therefore the weight of the portion $AD$ is $P$; and therefore the weight of the portion $DB$ is $Q$.

Since $C$ is the middle point of $AB$ the rod $AB$ will balance about $C$; and by Art. 107 if the part $AD$ be attached at its middle point $M$ to the lever $MN$, and the part $BD$ at its middle point $N$, the effect will be the same as before. Therefore in this case also the weights balance; that is $P$ at $M$ and $Q$ at $N$ balance.

109. *To deduce the Parallelogram of Forces from the Principle of the Lever.*

Let a force $P$ act along $Op$ and a force $Q$ along $Oq$; let $Or$ be the direction of their resultant.

From any point $C$ in the direction of the resultant draw $CA$ parallel to $Oq$ and $CB$ parallel to $Op$; also $CM$ perpendicular to $Op$ and $CN$ perpendicular to $Oq$.




If a force equal to the resultant of $P$ and $Q$ act along $rO$, it will with the forces $P$ and $Q$ keep a particle at $O$ in equilibrium. Suppose these forces applied by means of rods, connected at $O$; these rods will then be in equilibrium. Imagine the point $C$ on the rod $Or$ to become fixed; this will not disturb the equilibrium. The system can still turn round $C$, and it will do so unless the moments round $C$ are equal and contrary. Thus if there is equilibrium we must have, by the Principle of the Lever,

$$P \times CM = Q \times CN.$$

Therefore
$$\frac{P}{Q} = \frac{CN}{CM}$$

$$= \frac{CB}{CA}, \text{ by Euclid VI. 4, } = \frac{OA}{OB}.$$

Thus the forces $P$ and $Q$ are proportional to the sides $OA$ and $OB$ of the parallelogram $OACB$, and the *diagonal* $OC$ *represents the direction of their resultant.*

This demonstrates the Parallelogram of Forces so far as relates to the *direction* of the resultant; then as in Art. 49 we can demonstrate it also for the *magnitude* of the resultant.

## EXAMPLES. VII.

1. $ABCD$ is a square; a force of 3 lbs. acts from $A$ to $B$, a force of 4 lbs. from $B$ to $C$, and a force of 5 lbs. from $C$ to $D$: if the centre of the square be fixed, find the force which acting along $AD$ will keep the square in equilibrium.

2. The length of a horizontal lever is 12 feet, and the balancing weights at its ends are 3 lbs. and 6 lbs. respectively: if each weight be moved 2 feet from the end of the lever, find how far the fulcrum must be moved for equilibrium.

3. If the forces at the ends of the arms of a horizontal lever be 8 lbs. and 7 lbs., and the arms 8 inches and 9 inches respectively, find at what point a force of 1 lb. must be applied at right angles to the lever to keep it at rest.

4. The arms of a lever are inclined to each other: shew that the lever will be in equilibrium with equal weights suspended from its extremities, if the point midway between the extremities be vertically above or vertically below the fulcrum.

5. A weight $P$ suspended from one end of a lever without weight is balanced by a weight of 1 lb. at the other end of the lever; when the fulcrum is removed through half the length of the lever it requires 10 lbs. to balance $P$: determine the weight of $P$.

6. A rod capable of turning round one end, which is fixed, is kept at rest by two forces acting at right angles to the rod; the greater force is 6 lbs. and the distance between the points of application of the forces is half the distance of the greater force from the fixed end: find the smaller force. Shew that if any force be added to the smaller force, a force half as large again must be added to the greater force in order to preserve equilibrium.

7. $ABC$ is a triangle without weight, having a right angle at $C$, and $CA$ is to $CB$ as 4 is to 3; the triangle is fixed at $C$, and two forces $P$ and $Q$ acting at $A$ and $B$ in directions at right angles to $CA$ and $CB$ keep it at rest: find the ratio of $P$ to $Q$.

8. In Example 7 if the force $P$ act at $A$ at right angles to $AC$, and the force $Q$ act at $B$ at right angles to $BA$, find the ratio of $P$ to $Q$ in order that the triangle may be at rest.

9. The lower end $B$ of a rigid rod without weight 10 feet long is hinged to an upright post, and its other end $A$ is fastened by a string 8 feet long to a point $C$ vertically above $B$, so that $ACB$ is a right angle. If a weight of one ton be suspended from $A$ find the tension of the string.

10. $ABC$ is a bent lever without weight of which $B$ is the fulcrum; weights $P, Q$ suspended from $A, C$ respectively are in equilibrium when $BC$ is horizontal; weights $P$, $2Q$ similarly suspended are in equilibrium when $AB$ is horizontal: shew that the angle $ABC = 135^0$.

11. A triangle can turn in its own plane round a point which coincides with the centre of the inscribed circle; forces acting along the sides keep the triangle in equilibrium: shew that one of the forces is equal to the sum of the other two.

12. A string passes through a small heavy ring, and the ends of the string are attached to the ends of a lever without weight: shew that when the system is in equilibrium the ring is vertically under the fulcrum.

13. Forces $P$, $Q$, $R$, $S$ act at the middle points of a rhombus, outwards, in the plane of the rhombus, at right angles to the sides: find the condition of equilibrium when the rhombus can only move in its own plane round the point of intersection of its diagonals: find also the conditions of equilibrium when the rhombus is free.

14. Two forces $P$ and $Q$ acting at a point $O$ have a resultant $R$; any straight line meets the directions of $P$, $Q$, $R$ at $A$, $B$, $C$ respectively: shew that

$$\frac{P}{OA \cdot BC} = \frac{Q}{OB \cdot AC} = \frac{R}{OC \cdot AB}.$$

15. A rod $AB$ without weight 14 inches long is suspended by two strings from a peg $C$; the string $AC$ is 15 inches long, and the string $BC$ is 13 inches long; 130 lbs. is suspended from $A$, and 52 lbs. from $B$: when the whole is in equilibrium find the tensions of the strings.

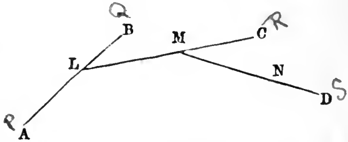
## VIII. *Centre of Parallel Forces.*

110. Suppose we have two parallel forces acting respectively at two points; we know that their resultant is equal to the algebraical sum of the forces, and is parallel to them, and that it may be supposed to act at a definite point on the straight line which passes through the two points. See Arts. 60 and 61. Moreover this definite point remains the same however the direction of the two forces be changed, so long as they remain parallel. This point is called the *centre* of the two parallel forces. Hence we adopt the following definition:

*The centre of a system of parallel forces is the point at which the resultant of the system may be supposed to act, whatever may be the direction of the parallel forces.*

111. *To find the resultant and the centre of any system of parallel forces.*

Let the parallel forces be $P$, $Q$, $R$, $S$, acting at the points $A$, $B$, $C$, $D$, respectively.




Join $AB$, and divide it at $L$, so that $AL$ may be to $LB$ as $Q$ is to $P$; then the resultant of $P$ at $A$ and $Q$ at $B$ is $P+Q$ parallel to them, at $L$.

Join $LC$, and divide it at $M$, so that $LM$ may be to $MC$ as $R$ is to $P+Q$; then the resultant of $P+Q$ at $L$ and $R$ at $C$ is $P+Q+R$ parallel to them, at $M$.

Join $MD$, and divide it at $N$, so that $MN$ may be to $ND$ as $S$ is to $P+Q+R$; then the resultant of $P+Q+R$ at $M$ and $S$ at $D$ is $P+Q+R+S$ parallel to them, at $N$.

Thus we have found the resultant and the centre of *four* parallel forces; and in the same way we may proceed whatever be the number of the forces.

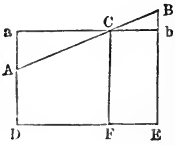
112. In the diagram and language of the preceding Article we have implied that the forces are all *like:* it is easy to make the slight modifications which are required when this is not the case. We may, if we please, form two groups, each consisting of like parallel forces, and obtain the resultant and the centre of each group; and then by Art. 61 deduce the resultant and the centre of the whole system of parallel forces.

We shall always obtain finally a single resultant and a definite centre, except in the case where the algebraical sum of all the forces is zero; and then either the forces are in equilibrium or they form a couple: see Art. 95.

Suppose that a system of parallel forces is formed into two groups in the manner just indicated; then if the resultant of one group is equal to the resultant of the other, and the centres of the two groups coincide, the whole system is in equilibrium. And conversely, if the whole system is in equilibrium the resultant of one group must be equal to the resultant of the other, and the centres of the two groups must coincide.

113. We have thus shewn how to determine *geometrically* the position of the centre of a system of parallel forces: we shall now shew how we may attain the same end by the aid of algebraical formulæ.

114. *The distances of the points of application of two parallel forces from a straight line being given, to determine the distance of the centre of the parallel forces from that straight line; the straight line and the points being all in one plane.*




First let $A$ and $B$ be the points of application of two *like* parallel forces, $P$ and $Q$; their resultant is $P+Q$, parallel to them, and it may be supposed to act at the point $C$, which is such that

$$\frac{P}{Q}=\frac{CB}{CA}.$$

Let $AD$, $BE$, $CF$ be perpendiculars from $A$, $B$, $C$ on any straight line which is in a plane containing $A$ and $B$. Let $AD=p$, $BE=q$, $CF=r$: then we have to find the value of $r$, supposing the values of $p$ and $q$ to be known.

Through $C$ draw $aCb$ parallel to $DFE$ meeting $AD$ and $BE$ at $a$ and $b$ respectively. Then

$$\frac{CB}{CA}=\frac{Bb}{Aa}, \text{ by Euclid VI. 4;}$$

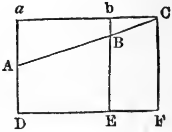
thus $$\frac{P}{Q}=\frac{Bb}{Aa}=\frac{BE-Eb}{Da-DA}=\frac{q-r}{r-p};$$

therefore $$P(r-p)=Q(q-r);$$

therefore $$(P+Q)r=Pp+Qq;$$

thus $$r=\frac{Pp+Qq}{P+Q}.$$

Next let $A$ and $B$ be the points of application of two *unlike* parallel forces $P$ and $Q$. Suppose $Q$ the greater. Then using the same construction and notation as before, we have




$$\frac{P}{Q}=\frac{CB}{CA}=\frac{Bb}{Aa};$$

thus $$\frac{P}{Q}=\frac{Eb-EB}{Da-DA}=\frac{r-q}{r-p};$$

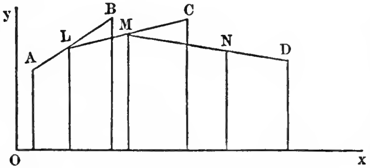
therefore $$P(r-p)=Q(r-q);$$

thus $$r=\frac{Qq-Pp}{Q-P}.$$

It will be observed that the result in the second case can be deduced from that in the first case by changing $P$ into $-P$.

115. *The distances of the points of application of any number of parallel forces from a straight line being given, to determine the distance of the centre of the parallel forces from that straight line, the straight line and the points being all in one plane.*

Let the parallel forces be $P$, $Q$, $R$, $S$ acting at the points $A$, $B$, $C$, $D$ respectively. Let $p$, $q$, $r$, $s$ be the distances of $A$, $B$, $C$, $D$ respectively from a straight line $Ox$ in the same plane as the points.




Join $AB$ and divide it at $L$, so that $AL$ may be to $LB$ as $Q$ is to $P$; then $L$ is the centre of $P$ at $A$ and $Q$ at $B$, and these forces are equivalent to $P+Q$ at $L$: let $l$ denote the distance of $L$ from $Ox$, then, by Art. 114,

$$l = \frac{Pp + Qq}{P + Q}.$$

Join $LC$ and divide it at $M$, so that $LM$ may be to $MC$ as $R$ is to $P+Q$; then $M$ is the centre of $P+Q$ at $L$ and $R$ at $C$, and these forces are equivalent to $P+Q+R$ at $M$: let $m$ denote the distance of $M$ from $Ox$, then, by Art. 114,

$$m = \frac{(P+Q)l + Rr}{P+Q+R} = \frac{Pp + Qq + Rr}{P+Q+R}.$$

Join $MD$ and divide it at $N$ so that $MN$ may be to $ND$ as $S$ is to $P+Q+R$; then $N$ is the centre of $P+Q+R$ at $M$ and $S$ at $D$, and these forces are equivalent to $P+Q+R+S$ at $N$: let $n$ denote the distance of $N$ from $Ox$, then, by Art. 114,

$$n = \frac{(P+Q+R)m + Ss}{P+Q+R+S} = \frac{Pp + Qq + Rr + Ss}{P+Q+R+S}.$$

Thus we have determined the distance from $Ox$ of the centre of *four* parallel forces; and in the same manner we may proceed whatever be the number of the forces.

The symmetrical form of the expression for $n$ should be noticed. We see that we shall obtain the same result in whatever order we combine the given forces, as we might have expected.

116. In the same way if the distances of $A$, $B$, $C$, and $D$ from a second straight line, as $Oy$, in the plane be given, we can deduce the distance of the centre of the parallel forces from the same straight line.

And when we know the distances of the centre from two straight lines in the plane we can determine the position of the centre; for the centre will be the point of intersection of straight lines parallel to $Ox$ and $Oy$, and at the respective distances from them which have been found.

117. In the figure and language of Art. 115 we have implied that the forces are all *like;* it is easy to make the slight modifications which are required when this is not the case. We may, if we please, form two groups, each consisting of like parallel forces, and obtain the centre of each group; and then by Art. 114 deduce the centre of the whole system of parallel forces.

The final result will be like that of Art. 115, the sign of those forces which act in one way being positive, and the sign of those which act in the other being negative.

118. We supposed in Art. 114 that the straight lines $AD$, $BE$, and $CF$ were all perpendicular to $DFE$. But this is not necessary; it is sufficient that these straight lines should be *all parallel.* And so also in Art. 115 the distances denoted by $p$, $q$, $r$, and $s$ need not necessarily be measured perpendicularly to the straight line $Ox$; it is sufficient that they should all be measured in parallel directions.

119. It is easy to extend our investigation to the case in which the points of application of the parallel forces are not all in one plane.

In the fundamental investigation of Art. 114 we may suppose $AD$, $CF$, and $BE$ to be the distances of $A$, $B$, and $C$, not from a *given straight line* but from a *given plane;* either perpendicular distances or distances measured parallel to a given straight line. Then, as in Arts. 114 and 115, if we know the distances of the points of application of the parallel forces from a given plane, we can obtain the distance of the centre of the parallel forces from that plane.

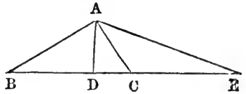
120. The weight of a body may be considered to be the aggregate of the weights of the particles which compose the body. The weights of these particles form a system of *like* parallel forces, and such a system always has a *centre;* see Art. 111. The centre of the parallel forces which consist of the weights of the particles of a body is called *the centre of gravity* of the body.

Thus the centre of gravity is a particular case of the centre of parallel forces; but it is found convenient to give especial attention to this particular case, and accordingly we shall consider it in the next Chapter. It will be observed that the theory of the centre of gravity is rather simpler than the general theory of the centre of parallel forces, because the weights of the particles of a body are all *like* forces, and thus we shall not have to consider the second case of Art. 114.

121. The following examples contain an interesting result.

(1) ABC *is a triangle; parallel forces act at* B *and* C, *each proportional to the opposite side of the triangle: determine the position of the centre of the parallel forces.*

First let the forces be *like.* In $BC$ take $D$ so that $BD$ is to $DC$ as the force at $C$ is to the force at $B$, that is as $AB$ is to $AC$; then $D$ is the centre of the parallel forces.




Hence, by Euclid, VI. 3, the point $D$ is such that $AD$ bisects the angle $BAC$.

Next let the forces be *unlike*. Suppose $AB$ greater than $AC$. Then, proceeding as before, we find that the centre of the parallel forces is at $E$ on $BC$ produced, such that $AE$ bisects the angle between $AC$ and $BA$ produced. See Euclid, VI. A.

(2) *Parallel forces act at the angular points of a triangle, each force being proportional to the opposite side of the triangle: determine the position of the centre of the parallel forces.*

First let the forces be all *like*. By the preceding example $D$ is the centre of the parallel forces at $B$ and $C$; hence the centre of all the three parallel forces lies on the straight line $AD$ which bisects the angle $BAC$. Similarly the centre lies on the straight line which bisects the angle $ABC$, and on the straight line which bisects the angle $BCA$. Therefore the centre of all the parallel forces must coincide with the centre of the circle inscribed in the triangle $ABC$.

Next let the forces be not all *like*. Suppose that the forces at $B$ and $C$ are unlike, and the forces at $A$ and $C$ like. By example (1) the centre of all the three parallel forces must lie on the straight line which bisects the angle between $CA$ and $BA$ produced, and also on the straight line which bisects the angle $ABC$, and on the straight line which bisects the angle between $AC$ and $BC$ produced. Therefore the centre of all the parallel forces must coincide with the centre of the circle which touches $AC$, and $BA$ and $BC$ produced. See *Notes on Euclid*, Book IV.

## EXAMPLES. VIII.

1. A body is acted on by two parallel forces $2P$ and $5P$, applied in opposite directions, their lines of action being 6 inches apart: determine the magnitude and the line of action of a third force which will be such as to keep the body at rest.

2. Parallel forces $P$ and $Q$ act at two adjacent corners of a parallelogram: determine the forces parallel to these which must act at the other corners, so that the

centre of the four parallel forces may be at the intersection of the diagonals of the parallelogram.

3. A rod without weight is a foot long; at one end a force of 2 lbs. acts, at the other end a force of 4 lbs., and at the middle point a force of 6 lbs., and these forces are all parallel and like: find the magnitude and the point of application of the single additional force which will keep the rod at rest.

4. Equal like parallel forces act at five of the angular points of a regular hexagon: determine the centre of the parallel forces.

5. Find the centre of like parallel forces of 7, 2, 8, 4, 6 lbs. which act in order at equal distances apart along a straight line.

6. The circumference of a circle is divided into $n$ equal parts, and equal like parallel forces act at all the points of division except one: find their centre.

7. Like parallel forces of 1, 2, and 3 lbs. act on a bar at distances 4, 6, and 7 inches respectively from one end: find their centre.

8. $ABC$ is a triangle; parallel forces $Q$ and $R$ act at $B$ and $C$ such that $Q$ is to $R$ as $\tan B$ is to $\tan C$: shew that their centre is at the foot of the perpendicular from $A$ on $BC$.

9. Parallel forces act at the angular points $A$, $B$, $C$ of a triangle, proportional to $\tan A$, $\tan B$, $\tan C$ respectively: shew that their centre is at the intersection of the perpendiculars drawn from the angles of the triangle on the opposite sides.

10. Parallel forces $P$, $Q$, $R$ act at the angular points $A$, $B$, $C$ of a triangle: shew that the perpendicular distance of their centre from the side $BC$ is

$$\frac{P}{P+Q+R} \times \frac{2 \text{ area of triangle}}{BC}.$$

11. Parallel forces $P$, $Q$, $R$ act at the angular points

$A$, $B$, $C$ of a triangle: shew that the distance of their centre from $BC$ measured parallel to $AB$ is

$$\frac{P \times AB}{P+Q+R}.$$

12. Parallel forces $P$, $Q$, $R$ act at the angular points $A$, $B$, $C$ of a triangle: determine the parallel forces which must act at the middle points of $BC$, $CA$, $AB$, so that the second system may have the same centre and the same resultant as the first system.

13. Like parallel forces of 3, 5, 7, 5 lbs. act at $A$, $B$, $C$, $D$, which are the angular points of a quadrilateral figure, taken in order: shew that the centre and the resultant will remain unchanged if instead of these forces we have acting at the middle points of $AB$, $BC$, $CD$, $DA$ respectively $P$, $10-P$, $4+P$, $6-P$ lbs., where $P$ may have any value.

14. Parallel forces $P$, $Q$, $R$ act at the angular points $A$, $B$, $C$ of a triangle; and their centre is at $O$: shew that

$$\frac{P}{\text{area of } BOC} = \frac{Q}{\text{area of } COA} = \frac{R}{\text{area of } AOB}.$$

15. Parallel forces act at the angular points $A$, $B$, $C$ of a triangle, proportional to $a \cos A$, $b \cos B$, $c \cos C$ respectively: shew that their centre coincides with the centre of the circumscribed circle.

16. Parallel forces $P$, $Q$, $R$, $S$ act at $A$, $B$, $C$, $D$; and

$$\frac{P}{\text{area of } BCD} = \frac{Q}{\text{area of } CDA} = \frac{R}{\text{area of } DAB} = \frac{S}{\text{area of } ABC}:$$

shew that their centre is at the intersection of $AC$ and $BD$.

17. Find the centre of equal like parallel forces acting at seven of the angular points of a cube.

18. Parallel forces $P$, $Q$, $R$, $S$ act at the angular points $A$, $B$, $C$, $D$ of a triangular pyramid: shew that the perpendicular distance of their centre from the face $BCD$ is

$$\frac{P}{P+Q+R+S} \times \frac{3 \text{ volume of pyramid}}{\text{area of } BCD}.$$

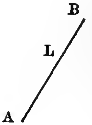
## IX. *Centre of Gravity.*

122. We begin with the following definition:

*The centre of gravity of a body or system of bodies is a point on which the body or system will balance in all positions, supposing the point to be supported, the body or system to be acted on only by gravity, and the parts of the body or system to be rigidly connected with the point.*

123. *To find the centre of gravity of two heavy particles.*

Let $A$ and $B$ be the positions of the two particles whose weights are $P$ and $Q$ respectively.



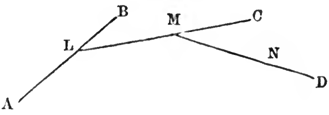

Join $AB$ and divide it at $L$, so that $AL$ may be to $LB$ as $Q$ is to $P$; then $L$ is the centre of gravity.

For, by Art. 60, the resultant of the weights $P$ and $Q$ acts through $L$; and therefore if $A$ and $B$ are connected by a rigid rod without weight the system will balance in every position when $L$ is supported.

As the resultant of $P$ and $Q$ is $P+Q$ the pressure on the point of support will be $P+Q$.

124. *To find the centre of gravity of any number of heavy particles.*

Let $A$, $B$, $C$, $D$ be the positions of particles whose weights are $P$, $Q$, $R$, $S$, respectively.



Join $AB$ and divide it at $L$, so that $AL$ may be to $LB$ as $Q$ is to $P$: then $L$ is the centre of gravity of $P$ at $A$ and $Q$ at $B$; and these weights produce the same effect as $P+Q$ at $L$. See Art. 123.

Join $LC$, and divide it at $M$, so that $LM$ may be to $MC$ as $R$ is to $P+Q$: then $M$ is the centre of gravity of $P+Q$ at $L$ and $R$ at $C$; and these weights produce the same effect as $P+Q+R$ at $M$.

Join $MD$, and divide it at $N$, so that $MN$ may be to $ND$ as $S$ is to $P+Q+R$: then $N$ is the centre of gravity of $P+Q+R$ at $M$ and $S$ at $D$; and these weights produce the same effect as $P+Q+R+S$ at $N$.

Then $N$ is the centre of gravity of the system; for the resultant of the weights passes through $N$, and therefore if the particles are connected with $N$ by rigid rods without weight the system will balance in every position when $N$ is supported.

Thus we have found the centre of gravity of *four* heavy particles; and in the same way we may proceed whatever be the number of the particles.

125. The investigation of the preceding Article shews that every system of heavy particles has a centre of gravity; for the construction there given is always possible.

We see that the resultant weight of a system of heavy particles always acts through the centre of gravity; so that the effect of the weight of the system is the same as if the whole weight were collected at the centre of gravity: this result might have been anticipated from the definition of the centre of gravity.

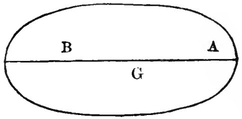
When we speak of a body or system balancing about its centre of gravity we shall not always explicitly say that the parts of the body or system are supposed to be rigidly connected with the centre of gravity; but this must always be understood.

126. *A body or a system of bodies cannot have more than one centre of gravity.*

For, if possible, suppose that the body or system of bodies has two centres of gravity, $G$ and $H$; and let $G$ and $H$ be brought into the same horizontal plane. Then when $G$ is supported the body or system balances, and therefore the vertical line in which the resultant weight of the body or system acts passes through $G$. Similarly the resultant weight acts through $H$. Thus a vertical line passes through two points which are in a horizontal plane; but this is absurd.

127. *If a body or a system of bodies balances itself on a straight line in every position, the centre of gravity of the body or system lies in that straight line.*

Let $AB$ be the straight line on which the body or system of bodies will balance in every position.




Suppose, if possible, that the centre of gravity is not in $AB$; let it be at $G$.

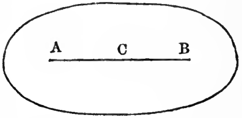
Place the body or system so that $AB$ is horizontal, and $G$ not in the vertical plane through $AB$. Suppose two points of the body or system situated on the straight line $AB$ to become fixed; this cannot disturb the equilibrium. The body or system has still the power of turning round $AB$ as an axis, and since it is in equilibrium the resultant weight of the system must pass through $AB$, by Art. 101. But this is impossible, because $G$ is neither vertically above nor vertically below $AB$.

Hence, the centre of gravity cannot be out of the straight line $AB$.

128. We shall now give two propositions which are almost immediately obvious, but which it is convenient to enunciate formally; and then we shall determine the position of the centre of gravity for some bodies of simple forms.

129. *Given the centres of gravity of two parts which compose a body or system of bodies, to find the centre of gravity of the whole body or system of bodies.*

Let $A$ and $B$ be the centres of gravity of the two parts; $P$ and $Q$ the respective weights of the parts.




Join $AB$ and divide it at $C$, so that $AC$ may be to $CB$ as $Q$ is to $P$: then $C$ is the centre of gravity of the whole body or system.

130. *Given the centre of gravity of part of a body or system of bodies, and also the centre of gravity of the whole body or system, to find the centre of gravity of the remainder.*

Let $A$ be the centre of gravity of the part, $C$ the centre of gravity of the whole; let $P$ be the weight of the part, and $W$ the weight of the whole.

Join $AC$ and produce it to $B$, so that $CB$ may be to $CA$ as $P$ is to $W-P$: then $B$ is the centre of gravity of the remainder.

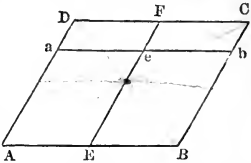
131. *To find the centre of gravity of a straight line.*

By a straight line here we mean a uniform material straight line, that is, a fine straight wire or rod, the breadth and thickness of which are constant and indefinitely small.

The centre of gravity of a uniform material straight line is at its middle point. For we may suppose the straight line to be made up of an indefinitely large number of equal particles. Take two of these which are equidistant from the middle point of the straight line; their centre of gravity is at the middle point. And since this is true for every such pair of particles the centre of gravity of the whole straight line is at the middle point of the straight line.

132. *To find the centre of gravity of a parallelogram.*

By a parallelogram here we mean a uniform material parallelogram; that is, a thin slice or lamina of matter, the thickness of which is constant and indefinitely small.




Let $ABCD$ be the parallelogram. Bisect $AB$ at $E$, and $CD$ at $F$; join $EF$. Draw any straight line $aeb$ parallel to $AEB$, meeting $AD$, $EF$, $BC$, at $a$, $e$, $b$ respectively. Then $DFea$ and $FCbe$ are parallelograms, from which it will follow that $ae = eb$.

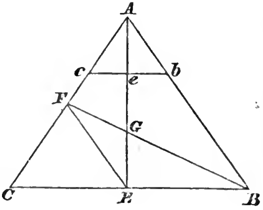
Suppose the parallelogram to be made up of indefinitely thin strips parallel to $AB$; the centre of gravity of each strip will be at its middle point by Art. 131; and will therefore be on the straight line $EF$. Hence the centre of gravity of the parallelogram is on the straight line $EF$.

Similarly the centre of gravity of the parallelogram is on the straight line which joins the middle points of $AD$ and $BC$.

Hence the centre of gravity of a parallelogram is at the intersection of the straight lines which join the middle points of opposite sides. This point coincides with the intersection of the diagonals of the parallelogram.

133. *To find the centre of gravity of a triangle.*

The meaning of the word *triangle* here is similar to that of the word *parallelogram* in the preceding Article.




Let $ABC$ be the triangle; bisect $BC$ at $E$; join $AE$. Draw any straight line $bec$ parallel to $BEC$, meeting $AB$, $AE$, $AC$ at $b$, $e$, $c$ respectively.

Then $\frac{be}{BE} = \frac{Ae}{AE}$, by Euclid, VI. 4;

similarly $$\frac{ce}{CE} = \frac{Ae}{AE};$$

therefore $$\frac{be}{BE} = \frac{ce}{CE}; \text{ therefore } \frac{be}{ce} = \frac{BE}{CE}.$$

But $BE = CE$; therefore $be = ce$.

Hence $e$ is the middle point of $bc$.

Suppose the triangle made up of indefinitely thin strips parallel to $BC$; the centre of gravity of every strip will be at its middle point by Art. 131, and will therefore be on the straight line $AE$. Hence the centre of gravity of the triangle is on the straight line $AE$.

In the same way if $AC$ be bisected at $F$ the centre of gravity of the triangle is on $BF$.

Hence the centre of gravity of the triangle must be at $G$, the point of intersection of $AE$ and $BF$.

Join $EF$; then $EF$ is parallel to $AB$, by Euclid, VI. 2;

therefore $$\frac{EG}{EF} = \frac{AG}{AB}, \text{ by Euclid, VI. 4};$$

therefore $$\frac{EG}{AG} = \frac{EF}{AB} = \frac{CE}{CB} = \frac{1}{2}.$$

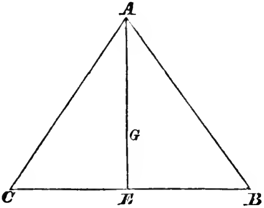
Thus $AG$ is twice $EG$, and therefore $AE$ is three times $EG$; that is, $EG$ is one third of $EA$.

Hence the centre of gravity of a triangle is determined by the following rule: Join any angular point with the middle point of the opposite side; the centre of gravity is on this straight line at one third of its length from the side.

The following statement will be very obvious, but it is useful to draw attention to it: Join an angle $A$ of a triangle with *any* point $L$ in the opposite side $BC$ or $BC$ produced; take $M$ in $LA$ so that $LM$ is one third of $LA$; and through $M$ draw a straight line parallel to $BC$: then the centre of gravity of the triangle is in this straight line.

134. *The centre of gravity of a triangle coincides with the centre of gravity of three equal heavy particles placed at the angular points of the triangle.*

Suppose equal heavy particles placed at the angular points of a triangle $ABC$.

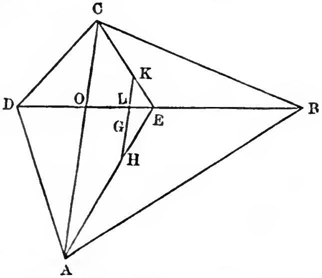



The centre of gravity of equal heavy particles at $B$ and $C$ is at $E$, the middle point of $BC$. Join $AE$ and divide it at $G$, so that $AG$ may be to $GE$ as 2 is to 1: then $G$ is the centre of gravity of equal heavy particles at $A$, $B$, and $C$. And $G$ coincides with the point which was found in the preceding Article to be the centre of gravity of the triangle $ABC$.

135. The centre of gravity of any plane rectilineal figure may be obtained in the following way: divide the figure into triangles, find the centre of gravity of each triangle, and then by successive applications of Art. 129 determine the centre of gravity of the proposed figure.

For example, suppose $ABCD$ to be any quadrilateral figure. Draw a diagonal $DB$ and bisect it at $E$; join $EA$ and $EC$.

Take $EH = \frac{1}{3} EA$, and $EK = \frac{1}{3} EC$. Then $H$ is the centre of gravity of the triangle $ABD$, and $K$ is the centre of gravity of the triangle $BCD$.




Join $HK$ and divide it at $G$, so that $HG$ may be to $KG$ as the triangle $CBD$ is to the triangle $ABD$: then $G$ is the centre of gravity of the quadrilateral figure.

Draw the diagonal $AC$, and let $O$ be the point of intersection of the two diagonals; let $HK$ meet $BD$ at $L$. Then the triangle $CBD$ is to the triangle $ABD$ as $CO$ is to $AO$; thus

$$\frac{HG}{KG} = \frac{CO}{AO}$$

$$= \frac{KL}{HL}, \text{ by similar triangles ;}$$

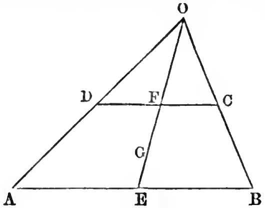
therefore $$\frac{HG}{HG+KG} = \frac{KL}{KL+HL}, \text{ that is } \frac{HG}{HK} = \frac{KL}{HK};$$

therefore $$HG = KL.$$

This gives a simple mode of determining $G$.

The case in which two sides of the quadrilateral are parallel may be specially noticed; to this we shall now proceed.

136. *To find the centre of gravity of a quadrilateral figure which has two sides parallel.*




Let $ABCD$ be a quadrilateral figure, having $AB$ parallel to $CD$: it is required to find the centre of gravity of the figure. Produce $AD$ and $BC$ to meet at $O$; let $E$ be the middle point of $AB$; join $OE$ meeting $CD$ at $F$. Then, as in Art. 133, we can shew that $DF = FC$, and that the centre of gravity of the quadrilateral is on $EF$.

The centre of gravity of the triangle $AOB$ is on $OE$, at a distance $\frac{2}{3} OE$ from $O$; and the centre of gravity of the triangle $DOC$ is on $OF$, at a distance $\frac{2}{3} OF$ from $O$. Let $G$ denote the centre of gravity of the quadrilateral $ABCD$.

By Art. 130, $$\frac{OG - \frac{2}{3}OE}{\frac{2}{3}OE - \frac{2}{3}OF} = \frac{\text{area of } DOC}{\text{area of } ABCD};$$

therefore $$\frac{OG - \frac{2}{3}OE}{OG - \frac{2}{3}OF} = \frac{\text{area of } DOC}{\text{area of } ABCD + \text{area of } DOC}$$

$$= \frac{\text{area of } DOC}{\text{area of } AOB}.$$

Now, by Euclid, VI. 19 and VI. 2,

$$\frac{\text{area of } DOC}{\text{area of } AOB}=\frac{OD^2}{OA^2}=\frac{OF^2}{OE^2};$$

therefore
$$\frac{OG-\frac{2}{3}OE}{OG-\frac{2}{3}OF}=\frac{OF^2}{OE^2}.$$

Hence $OG=\frac{2}{3}\frac{OE^3-OF^3}{OE^2-OF^2}=\frac{2}{3}\frac{OE^2+OE.OF+OF^2}{OE+OF}$;

therefore
$$FG=\frac{2}{3}\frac{OE^2+OE.OF+OF^2}{OE+OF}-OF$$

$$=\frac{2OE^2-OE.OF-OF^2}{3(OE+OF)}$$

$$=\frac{(OE-OF)(2OE+OF)}{3(OE+OF)}=\frac{FE}{3}.\frac{2OE+OF}{OE+OF}.$$

Since $\frac{OF}{OE}=\frac{CD}{AB}$, we have $\frac{2OE+OF}{OE+OF}=\frac{2AB+CD}{AB+CD}$.

Thus
$$FG=\frac{FE}{3}.\frac{2AB+CD}{AB+CD}.$$

Thus $FG$ is expressed in terms of the lengths of the two parallel sides and the distance of their middle points.

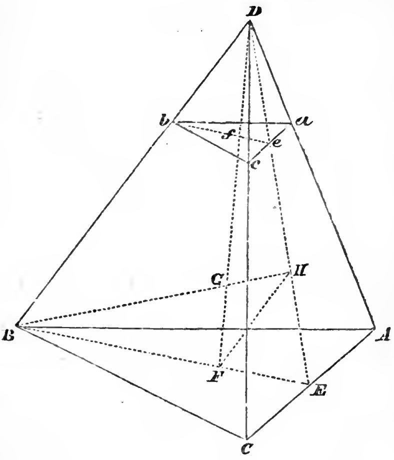
Or we may proceed thus. By drawing $DE$ and $CE$ the quadrilateral is divided into three triangles $ADE$, $BCE$, $CED$. Since these triangles are between the same parallels their areas will be in the proportion of their bases $AE$, $BE$, $CD$. The distances of the centres of gravity of these three triangles from $AB$, measured parallel to $EF$, will be respectively $\frac{1}{3}EF$, $\frac{1}{3}EF$, $\frac{2}{3}EF$. Thus, by Art. 115,

$$EG(AE+BE+CD)=\frac{1}{3}EF(AE+BE)+\frac{2}{3}EF.CD;$$

therefore
$$EG=\frac{EF}{3}.\frac{AB+2CD}{AB+CD}.$$

From this we get for $FG$, that is for $EF - EG$, the same value as before.

137. *To find the centre of gravity of a triangular pyramid.*




Let $ABC$ be the base, $D$ the vertex; bisect $AC$ at $E$; join $BE$ and $DE$. Take $F$ on $EB$ so that $EF = \frac{1}{3} EB$; then $F$ is the centre of gravity of the triangle $ABC$.

Join $FD$. From any point $b$ in $DB$ draw $ba$ and $bc$ parallel to $BA$ and $BC$ respectively.

Let $DF$ meet the plane $abc$ at $f$; join $bf$, and produce it to meet $DE$ at $e$.

Then, by similar triangles, $ae = ec$.

Also $$\frac{bf}{BF} = \frac{Df}{DF} = \frac{ef}{EF};$$

therefore $$\frac{bf}{ef}=\frac{BF}{EF}.$$

But $BF$ is twice $EF$; therefore $bf$ is twice $ef$; and therefore $f$ is the centre of gravity of the triangle $abc$.

Suppose the pyramid made up of indefinitely thin slices parallel to $ABC$; then, as we have just seen, the centre of gravity of every slice will be on the straight line $DF$. Hence the centre of gravity of the pyramid is on the straight line $DF$.

Again, take $H$ on $ED$ so that $EH=\frac{1}{3}ED$, and join $BH$. Then, as before, the centre of gravity of the pyramid is on $BH$.

Hence the centre of gravity of the pyramid must be at $G$, the point of intersection of $DF$ and $BH$.

Join $FH$; then $FH$ is parallel to $BD$ by Euclid, VI. 2.

Therefore $$\frac{HG}{HF}=\frac{BG}{BD}, \text{ by Euclid, VI. 4;}$$

therefore $$\frac{HG}{BG}=\frac{HF}{BD}=\frac{EF}{EB}=\frac{1}{3}.$$

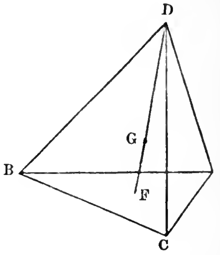
Thus $BG$ is three times $HG$, and therefore $BH$ is four times $HG$; that is, $HG$ is one fourth of $BH$.

Hence the centre of gravity of a triangular pyramid is determined by the following rule: Join any angular point with the centre of gravity of the opposite face; the centre of gravity of the pyramid is on this straight line at one fourth of its length from the face.

The following statement will be obvious: Join a vertex $D$ of a triangular pyramid with *any* point $L$ in the plane of the opposite face $ABC$; take $M$ in $LD$ so that $LM$ is one fourth of $LD$; and through $M$ draw a plane parallel to $ABC$: then the centre of gravity of the triangular pyramid is in this plane.

138. *The centre of gravity of a triangular pyramid coincides with the centre of gravity of four equal heavy particles placed at the angular points of the pyramid.*

Suppose equal heavy particles placed at the angular points of a pyramid $ABCD$.




The centre of gravity of equal heavy particles at $A$, $B$, and $C$, is at a point $F$ which coincides with the centre of gravity of the triangle $ABC$; see Art. 134. The effect of equal weights at $A$, $B$, and $C$ is the same as that of a triple weight at $F$. Join $DF$, and divide it at $G$ so that $DG$ may be to $GF$ as 3 is to 1: then $G$ is the centre of gravity of equal heavy particles at $A$, $B$, $C$, and $D$. And $G$ coincides with the point which was found in the preceding Article to be the centre of gravity of the pyramid $ABCD$.

139. *To find the centre of gravity of any pyramid having a plane rectilineal polygon for its base.*

The pyramid may be divided into triangular pyramids determined by drawing straight lines from any point on the base to all the angular points of the base, and to the vertex. The centre of gravity of every one of these triangular pyramids is in a plane which is parallel to the base, and at one fourth of its distance from the vertex. Hence the centre of gravity of the whole pyramid is in this plane.

Again, suppose the pyramid made up of indefinitely thin slices parallel to the base. It may be shewn, as in Art. 137, that the centre of gravity of every slice is on the straight line which joins the centre of gravity of the base of the whole pyramid with the vertex. Hence the centre of gravity of the whole pyramid is on this straight line.

Therefore the centre of gravity of the whole pyramid is on the straight line which joins the vertex with the centre

of gravity of the base, at one fourth of the length of this straight line from the base.

140. *To find the centre of gravity of a cone.*

A cone may be considered as a pyramid which has for its base a polygon with an indefinitely large number of sides. Hence the result obtained for a pyramid in Art. 139 holds for a cone. Therefore the centre of gravity of a cone is on the straight line which joins the vertex with the centre of gravity of the base, at one fourth of the length of this straight line from the base.

141. The principle of symmetry will often aid us in finding the position of the centre of gravity of a body.

A body is said to be *symmetrical* with respect to a plane when the body may be supposed to be made up of pairs of particles of equal size and weight, the two which form a pair being on opposite sides of the plane, equidistant from it and on the same perpendicular to it.

If a body be symmetrical with respect to a plane, that plane contains the centre of gravity of the body. For the weights of the two portions into which the plane divides the body are equal, and their centres of gravity are at equal distances from the plane on opposite sides of it: therefore the centre of gravity of the whole body is in the plane. See Art. 129.

142. If a body be symmetrical with respect to each of *two* planes, the centre of gravity will be in each of the planes, and therefore in the straight line in which they intersect. If a body be symmetrical also with respect to a third plane, the centre of gravity is in that plane; if the three planes have not a common line of intersection they will meet at a point, and this point will therefore be the centre of gravity of the body.

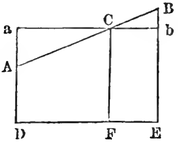
Take, for example, a sphere. Any plane passing through the centre of the sphere divides the sphere symmetrically, and so contains the centre of gravity: therefore the centre of the sphere is its centre of gravity.

143. The propositions respecting the centre of parallel forces, given in Arts. 114...119, are applicable to the centre of gravity, with the simplification which arises from the fact that the weights of particles are *like* parallel forces.

It will be convenient to repeat these propositions.

144. *The distances of two heavy particles from a straight line being given, to determine the distance of the centre of gravity of the particles from that straight line; the straight line and the particles being all in one plane.*

Let $A$ and $B$ be the positions of the particles; $P$ and $Q$ their respective weights.




Join $AB$, and divide it at $C$, so that $AC$ may be to $CB$ as $Q$ is to $P$: then $C$ is the centre of gravity of the particles.

Let $AD$, $BE$, $CF$ be perpendiculars from $A$, $B$, $C$ on any straight line which is in a plane containing $A$ and $B$. Let $AD=p$, $BE=q$, $CF=r$: then we have to find the value of $r$, supposing the values of $p$ and $q$ to be known.

Through $C$ draw $aCb$ parallel to $DFE$, meeting $AD$ and $BE$ at $a$ and $b$ respectively.

Then $$\frac{CB}{CA}=\frac{Bb}{Aa}, \text{ by Euclid, VI. 4;}$$

thus $$\frac{P}{Q}=\frac{Bb}{Aa}=\frac{BE-Eb}{Da-DA}=\frac{q-r}{r-p};$$

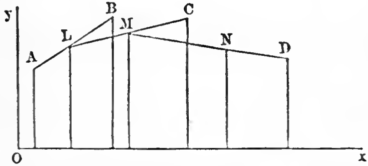
therefore $$P(r-p)=Q(q-r);$$

therefore $$(P+Q)r=Pp+Qq;$$

thus $$r=\frac{Pp+Qq}{P+Q}.$$

145. *The distances of any number of heavy particles in one plane from a straight line in the plane being given, to determine the distance of the centre of gravity of the system from that straight line.*

Let $A$, $B$, $C$, $D$ be the positions of particles whose weights are $P$, $Q$, $R$, $S$. Let $p$, $q$, $r$, $s$ be the distances of $A$, $B$, $C$, $D$ respectively from a straight line $Ox$ in the same plane.




Join $AB$, and divide it at $L$, so that $AL$ may be to $LB$ as $Q$ is to $P$; then $L$ is the centre of gravity of $P$ at $A$ and $Q$ at $B$, and these weights produce the same effect as $P+Q$ at $L$. Let $l$ denote the distance of $L$ from $Ox$, then, by Art. 144,

$$l=\frac{Pp+Qq}{P+Q}.$$

Join $LC$, and divide it at $M$, so that $LM$ may be to $MC$ as $R$ is to $P+Q$; then $M$ is the centre of gravity of $P+Q$ at $L$ and $R$ at $C$, and these weights produce the same effect as $P+Q+R$ at $M$. Let $m$ denote the distance of $M$ from $Ox$, then, by Art. 144,

$$m=\frac{(P+Q)\,l+Rr}{P+Q+R}=\frac{Pp+Qq+Rr}{P+Q+R}.$$

Join $MD$, and divide it at $N$, so that $MN$ may be to $ND$ as $S$ is to $P+Q+R$; then $N$ is the centre of gravity of $P+Q+R$ at $M$ and $S$ at $D$, and these weights produce the same effect as $P+Q+R+S$ at $N$. Let $n$ denote the distance of $N$ from $Ox$, then, by Art. 144,

$$n=\frac{(P+Q+R)\,m+Ss}{P+Q+R+S}=\frac{Pp+Qq+Rr+Ss}{P+Q+R+S}.$$

Thus we have determined the distance from $Ox$ of the centre of gravity of *four* heavy particles; and in the same manner we may proceed whatever be the number of heavy particles.

146. In the same way if the distances of $A$, $B$, $C$, and $D$ from a second straight line, as $Oy$, in the same plane be given, we can deduce the distance of the centre of gravity of the system from the same straight line.

And when we know the distance of the centre of gravity from two straight lines in the plane we can determine the position of the centre of gravity; for it will be at the point of intersection of straight lines parallel to $Ox$ and $Oy$ and at the respective distances from them which have been found.

It is easy to extend our investigation to the case in which the heavy particles are not all in one plane; see Art. 119. Thus if $p$, $q$, $r$, $s$ denote the distances from any *fixed plane* of particles whose weights are respectively $P$, $Q$, $R$, $S$, the value of $n$ in Art. 145 gives the distance of the centre of gravity of the particles from the same fixed plane. The distances may be either perpendicular distances, or distances measured parallel to any given straight line.

## EXAMPLES. IX.

1. If two triangles are on the same base, shew that the straight line which joins their centres of gravity is parallel to the straight line which joins their vertices.

2. A rod 3 feet long and weighing 4 lbs. has a weight of 2 lbs. placed at one end: find the centre of gravity of the system.

3. A quarter of a triangle is cut off by a straight line drawn parallel to one of the sides: find the centre of gravity of the remaining piece.

4. Find the centre of gravity of a uniform circular disc out of which another circular disc has been cut, the latter being described on a radius of the former as diameter.

5. If three men support a heavy triangular board at its three corners, compare the force exerted by each man.

6. Shew that the centre of gravity of a wire bent into a triangular shape coincides with the centre of the circle inscribed in the triangle formed by joining the middle points of the sides of the original triangle.

7. If the centre of gravity of a triangle be equidistant from two angular points of the triangle, the triangle must be isosceles.

8. If a straight line drawn from an angular point through the centre of gravity of a triangle be perpendicular to the opposite side, the triangle must be isosceles.

9. A triangle $ABC$ has the sides $AB$ and $BC$ equal; a portion $APC$ is removed such that $AP$ and $PC$ are equal: compare the distances of $P$ and $B$ from $AC$ in order that the centre of gravity of the remainder may be at $P$.

10. A heavy bar 14 feet long is bent into a right angle so that the lengths of the portions which meet at the angle are 8 feet and 6 feet respectively: shew that the distance of the centre of gravity of the bar so bent from the point of the bar which was the centre of gravity when the bar was straight, is $\frac{9\sqrt{2}}{7}$ feet.

11. If the centre of gravity of three heavy particles placed at the angular points of a triangle coincides with the centre of gravity of the triangle, the particles must be of equal weight.

12. Two equal uniform chains are suspended from the extremities of a straight rod without weight, which can turn about its middle point: find the position of the centre of gravity of the system, and shew that it is independent of the inclination of the rod to the horizon.

13. The middle points of two adjacent sides of a square are joined and the triangle formed by this straight line and the edges is cut off: find the centre of gravity of the remainder of the square.

14. If $n$ equal weights are to be suspended from a horizontal straight line by separate strings, and a given length $l$ of string is to be used, determine the distance of the centre of gravity of the weights from the straight line.

15. If the sides of a triangle be 3, 4, and 5 feet, find the distance of the centre of gravity from each side.

16. A piece of uniform wire is bent into the shape of an isosceles triangle; each of the equal sides is 5 feet long, and the other side is 8 feet long: find the centre of gravity.

17. Find the centre of gravity of the figure consisting of an equilateral triangle and a square, the base of the triangle coinciding with one of the sides of the square.

18. Two straight rods without weight each four feet long, are loaded with weights 1 lb., 3 lbs., 5 lbs., 7 lbs., 9 lbs. placed in order a foot apart: shew how to place one of the rods across the other, so that both may balance about a fulcrum at the middle point of the other.

19. A rod of uniform thickness is made up of equal lengths of three substances, the densities of which taken in order are in the proportion of 1, 2, and 3: find the position of the centre of gravity of the rod.

20. A table whose top is in the form of a right-angled isosceles triangle, the equal sides of which are three feet in length, is supported by three vertical legs placed at the corners; a weight of 20 lbs. is placed on the table at a point distant fifteen inches from each of the equal sides: find the resultant pressure on each leg.

21. In the diagram of Art. 136 shew that if $AC$ and $BD$ be joined, intersecting at $S$, then $S$ is on $FE$: shew also that $SG$ is equal to two thirds of the difference between $SE$ and $SF$.

## *X. Properties of the Centre of Gravity.*

147. *When a body is suspended from a point round which it can move freely it will not rest unless its centre of gravity be in the vertical line passing through the point of suspension.*

For the body is acted on by two forces, namely its own weight in a vertical direction through the centre of gravity, and the force arising from the fixed point. The body will not rest unless these two forces are equal and opposite. Therefore the centre of gravity must be in the vertical line which passes through the point of suspension.

148. The preceding Article suggests an experimental method of determining the centre of gravity of a body which may sometimes be employed. Let a body be suspended from a point about which it can turn freely, and let the direction of the vertical line through the point of suspension be determined. Again, let the body be suspended from another point so as to hang in a different position, and let the direction of the vertical line through the point of suspension be determined. The centre of gravity is in each of the two determined straight lines, and is therefore at their point of intersection.

149. *When a body can turn freely round an axis which is not vertical, it will not rest unless the centre of gravity be in the vertical plane passing through the axis.*

The weight of the body may be supposed to act at the centre of gravity. Resolve it into two components at right angles to each other, one component being parallel to the axis. The component parallel to the axis will not produce nor prevent motion round the axis; but the other component will set the body in motion round the axis, unless the centre of gravity be in the vertical plane passing through the axis.

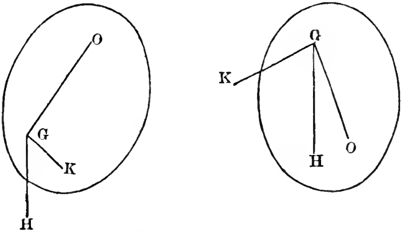
150. A body which is suspended from a fixed point by means of a *string* will not rest unless its centre of gravity be *below* the fixed point to which the string is fastened. But a body which can turn freely round a fixed point *rigidly* connected with it may rest with its centre of gravity either vertically *above* or vertically *below* the fixed

point. And in like manner when a body can turn freely round a fixed axis which is not vertical it may rest with its centre of gravity either *above* or *below* the axis. There is an important difference between the two positions of equilibrium, which is shewn by the following proposition.

151. *When a body which can turn freely round a fixed point is in equilibrium, if it be slightly displaced it will tend to return to its position of equilibrium or to recede from it according as the centre of gravity is below or above the fixed point.*

This may be taken as an experimental fact; or it may be established thus:

Let $O$ be the fixed point, $G$ the centre of gravity of the body. Draw $GH$ vertically downwards. The weight of




the body acts along $GH$; resolve it into two components at right angles to each other, one along the straight line which joins the centre of gravity with the fixed point: let $GK$ be the direction of the other component.

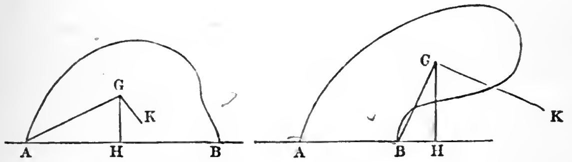
When $G$ is nearly below $O$ the former component acts along $OG$; and thus the latter obviously tends to move the body *towards* the position in which $G$ is vertically below $O$.

When $G$ is nearly *above* $O$ the former component acts along $GO$; and thus the latter obviously tends to move the body *away from* the position in which $G$ is vertically above $O$.

In the former case the body is said to be in *stable* equilibrium, and in the latter case in *unstable* equilibrium.

152. *When a body is placed on a horizontal plane it will stand or fall according as the vertical line drawn through its centre of gravity passes within or without the base.*

Let $G$ be the centre of gravity of a body. Let the vertical line through $G$ cut the horizontal plane on which the body stands at $H$. Let any horizontal straight line be drawn through $H$, and let $AB$ be that portion of it which is within the base of the body.




First suppose $H$ to be *between* $A$ and $B$.

No motion can take place round $A$. For the weight of the body acts vertically downwards at $G$; and it may be resolved into two components, one along $GA$, and the other at right angles to $GA$. The former component has no tendency to produce motion round $A$. The latter component tends to turn $G$ round $A$ in the direction $GK$; now if this motion could take place such a point as $B$ would turn round $A$ in a like direction, but this is prevented by the resistance of the plane at $B$: therefore $G$ cannot move round $A$.

Similarly no motion can take place round $B$; therefore the body cannot fall over either at $A$ or at $B$.

Next suppose $H$ not to be between $A$ and $B$: let it be on $AB$ produced through $B$.

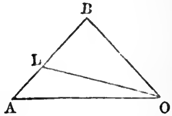
Then, as before, no motion will take place round $A$. But motion will take place round $B$; for the tendency of the component of the weight at right angles to $GB$ is to move $G$ round $B$ in the direction $GK$; and there is nothing to prevent this motion.

153. The sense in which the word *base* is used in the preceding proposition may require some explanation. The portions of surface common to the body and the horizontal plane may form one undivided area, or may consist of various separate areas; a smooth brick placed on a smooth horizontal plane will exemplify the former case, and a chair will exemplify the latter case. Moreover these areas may be indefinitely small, that is, may be mere points.

The boundary of the *base* for the purposes of the preceding proposition must be determined thus: let a polygon be formed by straight lines joining the points of contact, in such a manner as to include all the points of contact, and to have no *re-entrant* angle. See *Notes on Euclid*, I. 32.

154. *Forces are represented in direction by the straight lines drawn from any point to a system of heavy particles, each force being equal to the product of the length of the straight line into the weight of the corresponding particle: to shew that the resultant force is represented in direction by the straight line drawn from the point to the centre of gravity of the particles, and is equal to the product of the length of this straight line into the sum of the weights.*

First, let there be *two* heavy particles. Suppose $P$ and $Q$ their weights; $A$ and $B$ their respective positions. Let $L$ be their centre of gravity.

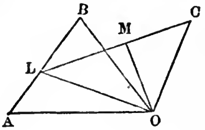



Let $O$ be any point; and suppose there are two forces, namely $P \times OA$ along $OA$, and $Q \times OB$ along $OB$.

The force $P \times OA$ along $OA$ may be resolved into $P \times OL$ along $OL$, and $P \times LA$ parallel to $LA$. The force $Q \times OB$ along $OB$ may be resolved into $Q \times OL$ along $OL$ and $Q \times LB$ parallel to $LB$.

The two forces $P \times LA$ and $Q \times LB$ are thus equal and opposite, and therefore balance each other. Hence the resultant is $(P+Q)\ OL$ along $OL$.

Next, let there be *three* heavy particles. Suppose $P$, $Q$, $R$ their weights; $A$, $B$, $C$ their respective positions. Let $L$ be the centre of gravity of $P$ and $Q$; and $M$ the centre of gravity of $P$, $Q$, and $R$.




Let $O$ be any point, and suppose there are three forces, namely, $P \times OA$ along $OA$, $Q \times OB$ along $OB$, and $R \times OC$ along $OC$.

By what has been already shewn, these forces are equivalent to $(P+Q)\ OL$ along $OL$, and $R \times OC$ along $OC$: and these again are equivalent to $(P+Q+R)\ OM$ along $OM$.

If there be a fourth heavy particle, of weight $S$, at a point $D$, there are four forces which, by what has been shewn, are equivalent to $(P+Q+R)\ OM$ along $OM$, and $S \times OD$ along $OD$: and these again are equivalent to $(P+Q+R+S)\ ON$ along $ON$, where $N$ is the centre of gravity of the four heavy particles.

In this manner the proposition may be established, whatever be the number of heavy particles.

If the point at which the directions of the forces meet coincides with the centre of gravity of the system of heavy particles, the resultant is zero; that is, the forces are then in equilibrium.

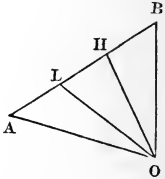
155. *Forces are represented in magnitude and direction by straight lines drawn from any point: to shew that the resultant force is represented in direction by the straight line drawn from this point to the centre of gravity of a system of equal particles situated at the other extremities of the straight lines, and is equal to the product of this straight line into the number of particles.*

This is a particular case of the proposition of the preceding Article, obtained by supposing all the heavy particles there to be of equal weight.

If the point at which the directions of the forces meet coincides with the centre of gravity of the system of equal

particles, the resultant is zero; that is, the forces are then in equilibrium.

156. *The sum of the products of the weight of each particle of a system of heavy particles into the square of its distance from any point exceeds the sum of the products of the weight of each particle into the square of its distance from the centre of gravity by the product of the sum of the weights into the square of the distance between the point and the centre of gravity.*




First, let there be *two* heavy particles. Suppose $P$ and $Q$ their weights; $A$ and $B$ their respective positions. Let $L$ be their centre of gravity.

Let $O$ be any point: then shall

$$P \times OA^2 + Q \times OB^2$$
$$= P \times AL^2 + Q \times BL^2 + (P+Q)\, OL^2.$$

Let $OH$ be the perpendicular from $O$ on $AB$.

By Euclid, II. 12, 13,

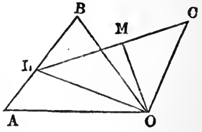
$$OA^2 = AL^2 + OL^2 + 2AL \,.\, LH,$$
$$OB^2 = BL^2 + OL^2 - 2BL \,.\, LH;$$

therefore

$$P \times OA^2 + Q \times OB^2 = P \times AL^2 + Q \times BL^2 + (P+Q)\, OL^2,$$

for $P \times AL = Q \times BL$, by the nature of the centre of gravity.

In the figure $H$ falls between $L$ and $B$; the demonstration is essentially the same for every modification of the figure.




Next, let there be *three* heavy particles. Suppose $P$, $Q$, $R$ their weights; $A$, $B$, $C$ their respective positions. Let $L$ be the centre of gravity of $P$ and $Q$, and $M$ the centre of gravity of $P$, $Q$, and $R$.

Then we have, by three applications of the result already obtained,

$$
\begin{aligned}
P \times OA^2 + Q \times OB^2 + R \times OC^2 \\
&= P \times LA^2 + Q \times LB^2 + (P+Q)\,OL^2 + R \times OC^2 \\
&= P \times LA^2 + Q \times LB^2 + (P+Q)\,ML^2 + R \times MC^2 \\
&\qquad + (P+Q+R)\,OM^2 \\
&= P \times MA^2 + Q \times MB^2 + R \times MC^2 \\
&\qquad + (P+Q+R)\,OM^2.
\end{aligned}
$$

Similarly, by three applications of results already obtained, we can shew that the proposition is true when there are *four* heavy particles: and so on universally.

157. *If the weight of each of a system of heavy particles be multiplied into the square of the distance of the particle from a given point, the sum of the products is least when the given point is the centre of gravity of the system.*

This follows immediately from the proposition of the preceding Article.

158. Examples may be proposed respecting the centre of gravity which do not involve any new mechanical conception, but are merely geometrical deductions.

For example, required the distances of the centre of gravity of a triangle from the three angular points in terms of the sides of the triangle.

Let $ABC$ denote a triangle, $D$ the middle point of $BC$, $G$ the centre of gravity. Then $G$ is in $AD$, and $AG = \frac{2}{3} AD$.

Now, by the *Appendix to Euclid*, Art. 1,

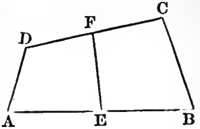
$$AB^2 + AC^2 = 2\,(AD^2 + BD^2);$$

therefore $AD^2 = \frac{1}{2}\left(AB^2 + AC^2 - \frac{1}{2}BC^2\right)$. And $AG^2 = \frac{4}{9} AD^2$;

therefore $AG^2 = \frac{2}{9}\left(AB^2 + AC^2 - \frac{1}{2}BC^2\right)$.

Similar expressions may be found for $BG^2$ and $CG^2$.

159. The theory of the centre of gravity will furnish us with indirect demonstrations of geometrical theorems. We will give an example.




Let $A$, $B$, $C$, $D$ be four points, which need not be all in the same plane; and let equal heavy particles be placed at these points. Let $E$ be the middle point of $AB$, and $F$ the middle point of $CD$. Then $E$ is the centre of gravity of the particles at $A$ and $B$, and $F$ is the centre of gravity of the particles at $C$ and $D$. Therefore the centre of gravity of the system is at the middle point of $EF$. In the same way the centre of gravity of the system is at the middle point of the straight line which joins the middle points of $AD$ and $BC$. But there is only one centre of gravity of the system; and hence we obtain the following result: *The straight lines which join the middle points of the opposite sides of any quadrilateral bisect each other.*

Similarly, from the process for finding the centre of gravity of a triangle, we might infer that *the straight lines which join the angular points of a triangle with the middle points of the opposite sides meet at a point.*

## EXAMPLES. X.

1. A square stands on a horizontal plane: if equal portions be removed from two opposite corners by straight lines parallel to a diagonal, find the least portion which can be left so as not to topple over.

2. Find the locus of the centres of gravity of all triangles on the same base and between the same parallels.

3. A portion of the surface of a heavy body is spherical, and the body is in equilibrium when any point of this portion is in contact with a horizontal plane: find the position of the centre of gravity of the body.

4. Given the base and the height of a triangle, construct it so that it may just rest in equilibrium with its base on a horizontal plane.

5. A quadrilateral lamina which has all its sides equal will be in equilibrium if its plane be vertical and any one of its sides on a horizontal plane.

6. Two weights $W$ and $2W$ are connected by a rod without weight, and also by a loose string which is slung over a smooth peg: compare the lengths of the string on each side of the peg when the weights have assumed their position of equilibrium.

7. If a number of right-angled triangles be described on the same straight line as hypotenuse, their centres of gravity all lie on a circle.

8. If the sides of a triangle be bisected, and the triangle formed by joining these points be removed, shew that the centre of gravity of the remainder will coincide with that of the whole triangle.

9. A round table stands on three legs placed on the circumference at equal distances: shew that a body whose weight is not greater than that of the table may be placed on any point of it without upsetting it.

10. $ABCD$ is a parallelogram having the angle $ABC = 60^\circ$, and the base $BC$ six inches in length: determine the greatest possible length of $AB$ if the figure is to stand on $BC$.

11. A heavy triangle is to be suspended by a string passing through a point on one side: determine the position of the point so that the triangle may rest with one side vertical.

12. A triangle obtuse-angled at $B$ is placed with its side $CB$ resting on a horizontal plane; a vertical straight line from $A$ meets the plane at $D$: shew that the triangle will stand or fall according as $BD$ is less or greater than $BC$.

13. The sides of a heavy triangle are 3, 4, 5 respectively: if it be suspended from the centre of the inscribed circle shew that it will rest with the shortest side horizontal.

14. The altitude of a right cone is $h$, and a diameter of the base is $b$; a string is fastened to the vertex and to a point on the circumference of the circular base, and is then put over a smooth peg: shew that if the cone rests with its axis horizontal the length of the string is $\sqrt{(h^2 + b^2)}$.

## XI. *The Lever.*

160. *Machines* are instruments used for communicating motion to bodies, for changing the motion of bodies, or for preventing the motion of bodies.

The most simple machines are called *Mechanical Powers;* by combining these, all machines, however complicated, are constructed. These simple Machines or Mechanical Powers are usually considered to be *seven* in number; namely the Lever, the Wheel and Axle, the Toothed Wheel, the Pully, the Inclined Plane, the Wedge, and the Screw.

We shall investigate the conditions of *equilibrium* of the Mechanical Powers; that is, we shall suppose these simple machines employed to *prevent* motion. We shall in every case have two forces which balance each other by means of a machine; one force for the sake of distinction is called the *Power*, and the other the *Weight:* we shall find that in every case for equilibrium the Power must bear to the Weight a certain ratio which depends on the nature of the machine.

We shall assume, unless the contrary is expressly stated, that the parts of the machine are smooth and without weight.

In the present Chapter we shall consider the Lever.

161. The *Lever* is an inflexible rod moveable, in one plane, about a point in the rod which is called the *fulcrum.* The parts of the Lever between the fulcrum and the points of application of the Power and the Weight are called the *arms* of the Lever. When the arms are in the same straight line the lever is called a *straight Lever;* in other cases it is called a *bent Lever.* The plane in which the Lever can move may be called the *plane of the Lever.* The forces which act on the Lever are supposed to act in the plane of the Lever.

162. Levers are sometimes divided into three classes, according to the positions of the points of application of the Power and the Weight with respect to the fulcrum.

In the first class the Power and the Weight act on *opposite* sides of the *fulcrum.*

In the second class the Power and the Weight act on the *same* side of the fulcrum, the *Weight* being the *nearer* to the fulcrum.

In the third class the Power and the Weight act on the *same* side of the fulcrum, the *Power* being the *nearer* to the fulcrum.

Thus we may say briefly that the three classes have respectively the Fulcrum, the Weight, and the Power in the middle position.

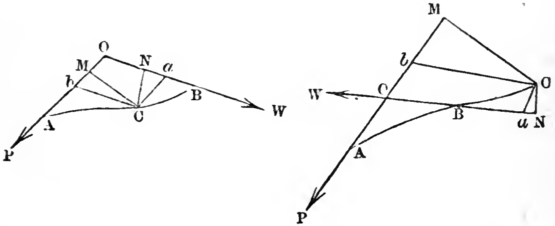
163. The following are examples of Levers of the first class: a crow-bar used to raise a heavy weight, a poker used to raise coals in a grate, the brake of a pump. In scissors, shears, nippers, and pincers we have examples of a double Lever of the first class.

The oar of a boat furnishes an example of a Lever of the second class. The fulcrum is at the blade of the oar in the water; the Power is applied by the hand; the Weight is applied at the rowlock. A wheel-barrow in use will also serve as an example. A pair of nut-crackers is a double Lever of the second class.

A pair of tongs used to hold a coal is a double Lever of the third class. The fulcrum is the pivot on which the two parts of the instrument turn; the Power is the pressure applied by the hand; the Weight is the resistance of the coal at the end of the tongs. An example of the third class of Lever is seen in the human fore-arm employed to raise an object taken in the hand. The fulcrum is at the elbow; the Power is exerted by a muscle which comes from the upper part of the arm, and is inserted in the fore-arm near the elbow; the Weight is the object raised in the hand.

164. The necessary and sufficient condition for the equilibrium of two forces on the Lever is that their moments round the fulcrum should be equal in magnitude but of opposite kinds. This has been already demonstrated; see Art. 102. But on account of the importance of the principle of the Lever we shall give a separate investigation.

165. *When there is equilibrium on the Lever the Power is to the Weight as the length of the perpendicular from the fulcrum on the direction of the Weight is to the length of the perpendicular from the fulcrum on the direction of the Power.*




Let $ACB$ or $ABC$ be a Lever, $C$ being the fulcrum. Let forces $P$ and $W$ act at $A$ and $B$ respectively and keep the Lever in equilibrium.

Let the directions of $P$ and $W$ meet at $O$. Then the resultant of $P$ and $W$ will be some force which may be supposed to act at $O$; and this resultant must pass through $C$, since the Lever is in equilibrium. Hence $OC$ is the direction of the resultant of $P$ and $W$.

Draw $Ca$ parallel to $OA$, and $Cb$ parallel to $OB$, to meet $OB$ and $OA$ respectively; and draw $CM$ perpendicular to $OA$, and $CN$ perpendicular to $OB$.

Then, by the Parallelogram of Forces,

$$\frac{P}{W} = \frac{Ca}{Cb};$$

and by the similar triangles $CNa$ and $CMb$,

$$\frac{Ca}{Cb} = \frac{CN}{CM};$$

therefore $$\frac{P}{W} = \frac{CN}{CM}.$$

Conversely, if $\frac{P}{W} = \frac{CN}{CM}$, and $P$ and $W$ tend to turn the Lever in opposite directions, they will keep it in equilibrium.

For with the same construction we have

$$\frac{P}{W} = \frac{CN}{CM} = \frac{Ca}{Cb};$$

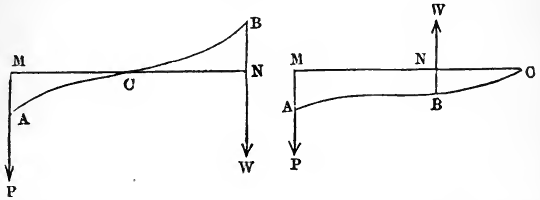
and therefore $OC$ is the direction of the resultant of $P$ and $W$; and since the resultant passes through $C$ the Lever will be kept in equilibrium.

166. There is no substantial difference in theory between the second and the third class of Levers considered in Art. 162; but there is considerable practical difference. For it follows from the condition of equilibrium of the Lever that in the second class the Power is less than the Weight, and in the third class the Power is greater than the Weight. Thus it is said that a mechanical *advantage* is gained by a Lever of the second class, and lost by a Lever of the third class.

The word *advantage* is used in a popular sense in the remark just made; more strictly the *advantage* of a machine may be defined as the ratio of the Weight to the Power when there is equilibrium.

167. In the investigation of Article 165 we assume that the directions of $P$ and $W$ will meet if produced; but the point of intersection may be at any distance from the fulcrum, so that we may readily admit that the result will hold even when the directions of $P$ and $Q$ are *parallel*. But it may be useful to give an investigation of this case.

Let $ACB$ or $ABC$ be a Lever, $C$ being the fulcrum. Let parallel forces $P$ and $W$ act at $A$ and $B$ respectively and keep the Lever in equilibrium.




Through $C$ draw a straight line perpendicular to the directions of the forces meeting them at $M$ and $N$ respectively.

Now by Arts. 60 and 61 the resultant of $P$ and $W$ is a force parallel to them at distances from them which are inversely proportional to them. But since the Lever is in equilibrium the resultant must pass through $C$; and therefore

$$\frac{P}{W}=\frac{CN}{CM}.$$

Conversely, if $\frac{P}{W}=\frac{CN}{CM}$, and $P$ and $W$ tend to turn the Lever in opposite directions, they will keep the Lever in equilibrium.

For since $\frac{P}{W}=\frac{CN}{CM}$, the resultant of $P$ and $W$ passes through $C$, and therefore the Lever will be kept in equilibrium.

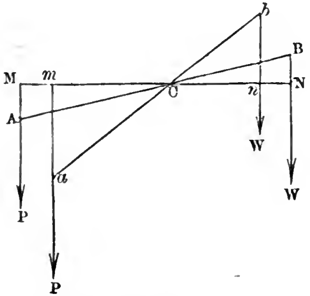
168. It appears from the foregoing Articles that equilibrium is maintained on the Lever by the aid of the fulcrum which supplies a force equal and opposite to the resultant of $P$ and $W$. Thus we see that the pressure on the fulcrum will be equal to the resultant of $P$ and $W$; if $P$ and $W$ are parallel this resultant is equal to their algebraical sum, in other cases it may be determined by the Parallelogram of Forces.

169. *If two weights balance each other on a straight Lever in any one position inclined to the vertical, they will balance each other in any other position of the Lever.*

Let $AB$ be the position of the Lever when the weights $P$ and $W$ balance each other; let $C$ be the fulcrum.

Let $ab$ be any other position of the lever in the same vertical plane.

Through $C$ draw a horizontal line, meeting the vertical lines which represent the lines of action of the weights at $M$ and $N$ and $m$ and $n$ respectively.




Now since $P$ and $W$ balance in the position $AB$ of the Lever,

$$\frac{P}{W} = \frac{CN}{CM}.$$

And by similar triangles,

$$\frac{CN}{CM} = \frac{CB}{CA} = \frac{Cb}{Ca} = \frac{Cn}{Cm};$$

therefore $$\frac{P}{W} = \frac{Cn}{Cm}.$$

Hence $P$ and $W$ will balance each other in the position $ab$ of the Lever.

## EXAMPLES. XI.

1. A weight of 5 lbs. hung from one extremity of a straight Lever balances a weight of 15 lbs. hung from the other: find the ratio of the arms.

2. Two weights of $3\frac{1}{3}$ lbs. and $4\frac{1}{3}$ lbs. are hung at the ends of a straight Lever whose length is 92 inches: find where the fulcrum must be for equilibrium.

3. Two weights which together weigh $6\frac{1}{2}$ lbs. are hung at the ends of a straight Lever and balance: if the fulcrum is four times as far from one end as from the other find each weight.

4. A Lever 7 feet long is supported in a horizontal position by props placed at its extremities: find where a weight of 28 lbs. must be placed so that the pressure on one of the props may be 8 lbs.

5. Two weights of 12 lbs. and 8 lbs. respectively at the ends of a horizontal Lever 10 feet long balance: find how far the fulcrum ought to be moved for the weights to balance when each is increased by 2 lbs.

6. If the pressure on the fulcrum be equivalent to a weight of 15 lbs., and the difference of the forces to a weight of 3 lbs., find the forces and the ratio of the arms at which they act.

7. A Lever is in equilibrium under the action of the forces $P$ and $Q$, and is also in equilibrium when $P$ is trebled and $Q$ is increased by 6 lbs.: find the magnitude of $Q$.

8. The pressure on the fulcrum is 12 lbs., and the distance of the fulcrum from the middle point of the Lever is one-twelfth of the whole length of the Lever: find the forces which acting on opposite sides of the fulcrum will produce equilibrium.

9. One force is four times as great as the other, and the forces are on the same side of the fulcrum, and the pressure is 9 lbs. on it: find the position and the magnitude of the forces.

10. $ABC$ is a straight weightless rod 9 inches long, placed between two pegs $A$ and $B$ which are 4 inches apart, so as to be kept horizontal by means of them and a weight of 10 lbs. hanging at $C$: find the pressures on the pegs.

11. A Lever bent at right angles, with the angle for fulcrum and having one arm double the other, has two weights hanging from its ends: if in the position of equilibrium the arms are equally inclined to the horizon compare the weights.

12. The pressure on the fulcrum is 3 lbs., and the sum of the forces 10 lbs.: find the distance of each from the fulcrum, if their distance apart be 2 feet.

13. If the pressure on the fulcrum be 5 lbs., and one of the weights be distant from the fulcrum one-sixth of the whole length of the Lever, find the weights, supposing them on opposite sides of the fulcrum.

14. If the arm of a cork compresser be 18 inches, and a cork be placed at a distance of one inch and a half from the fulcrum, find the pressure produced by a weight of twelve stone suspended from the handle.

15. If the fulcrum be between the two forces, and its distance from one of them be a third of the whole length of the Lever, shew that when the direction of either of the forces is reversed, the fulcrum must then be placed at three times its former distance from the same force.

16. Two forces of 2 lbs. and 4 lbs. act at the same point of a straight Lever on opposite sides of it, and keep it at rest, the less force being perpendicular to the Lever: determine the direction of the greater force, and the pressure on the fulcrum.

17. A weight of $P$ lbs. hangs from the end of a Lever 2 feet long, at the other end of which is a fulcrum, and the Lever is kept in equilibrium by such a force $Q$ that the fulcrum bears $\frac{2}{3}P$ lbs.: determine the magnitude of $Q$, and the point of its application.

18. If three weights $P$, $Q$, $S$ hang from the points $A$, $B$, $C$ of a straight Lever which balances about a fulcrum $D$, shew that

$$Q \times AB + S \times AC = (P + Q + S) \times AD.$$

19. $ABC$ is a straight Lever; the length of $AB$ is 7 inches, that of $BC$ is 3 inches; weights of 6 lbs. and 10 lbs. hang at $A$ and $B$, and an upward pressure of 6 lbs. acts at $C$: find the position of a fulcrum about which the Lever so acted on would balance, and determine the pressure on the fulcrum.

20. Weights of 6 lbs. and 4 lbs. hang at distances 2 inches and 6 inches respectively from the fulcrum of a Lever on the same side of it: find where a single force of 9 lbs. must be applied to support them so as to leave the least possible pressure on the fulcrum.

21. Shew that the proposition of Art. 169 holds when there are more than two weights if they are applied at points of one straight line passing through $C$.

22. $ACB$ is a bent Lever; the arms $CA$, $CB$ are straight, and inclined to one another at an angle of $135^0$. When $CA$ is horizontal a weight $P$ at $A$ just sustains a weight $W$ at $B$; and when $CB$ is horizontal the weight $W$ at $B$ requires a weight $Q$ at $A$ to balance it: find the ratio of $Q$ to $P$.

23. A Lever $ACB$ is bent at $C$, the fulcrum, and from $B$ a weight $Q$ is hung; when $P$ is hung at $A$ the Lever rests with $AC$ horizontal; but when $S$ is hung at $A$ then $CB$ becomes horizontal: shew that

$$CA : CB :: Q : \sqrt{(P . S)}.$$

24. A Lever is 5 feet long, and from its ends a weight is supported by two strings 3 feet and 4 feet long respectively: shew that the fulcrum must divide the Lever into two parts the ratio of which is that of 9 to 16, if there be equilibrium when the Lever is horizontal.

## XII. *Balances.*

170. The Lever is employed to determine the weights of substances; and under this character it is called a *Balance:* we shall now describe various forms of the Balance.

In the preceding Chapter we considered a Lever to be a rod *without weight;* but in practice a rod always has weight, and we shall accordingly attend to this fact in our investigations.

We shall first consider the Common Balance.

171. *The Common Balance.* The Common Balance consists of a beam with a scale suspended from each extremity; the beam can turn about a fulcrum which is *above* the centre of gravity of the beam, and therefore above the centre of gravity of the system formed by the beam, the scales, and the things which may be put in the scales. The arms of the beam must be of equal length, and the system should be in equilibrium with the beam horizontal when the scales are empty: if these conditions hold, the Balance is said to be *true*, if they do not hold, the Balance is said to be *false.*

The substance to be weighed is placed in one scale, and weights in the other until the beam remains at rest in the horizontal position. In this case, if the Balance be *true*, the weight of the substance is indicated by the weights which balance it. We may test whether the Balance is true, by observing whether the beam still remains at rest in the horizontal position when the contents of the scales are interchanged.

If the Balance be true and the contents of the two scales be made of unequal weight, the beam will not remain in the horizontal position, but after oscillating for a time will finally rest in some position inclined to the horizon.

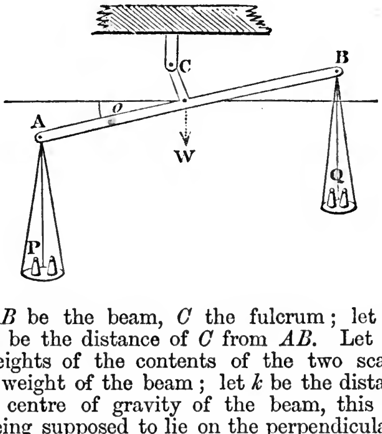
172. In the construction of a Balance, the following requisites should be satisfied:

(1) When loaded with equal weights the beam should be perfectly horizontal: that is, the Balance should be *true.*

(2) When the contents of the two scales differ in weight, even by a small quantity, the Balance should detect this difference: that is, the Balance should be *sensible.*

(3) When the Balance is disturbed, it should readily return to its state of rest: that is, the Balance should be *stable.*

173. *To find how the requisites of a good Balance may be satisfied.*




Let $AB$ be the beam, $C$ the fulcrum; let $AB=2a$, and let $h$ be the distance of $C$ from $AB$. Let $P$ and $Q$ be the weights of the contents of the two scales. Let $W$ be the weight of the beam; let $k$ be the distance from $C$ of the centre of gravity of the beam, this centre of gravity being supposed to lie on the perpendicular from $C$ on $AB$. Let $S$ be the weight of each scale, so that $P$ and $S$ act vertically through $A$, and $Q$ and $S$ act vertically through $B$. Let $\theta$ be the angle which the beam makes with the horizon when there is equilibrium.

The sum of the moments of the weights round $C$ will be zero when there is equilibrium, by Art. 86. Now the length of the perpendicular from $C$ on the line of action of $P$ and $S$ is $a \cos \theta - h \sin \theta$; the length of the perpendicular from $C$ on the line of action of $Q$ and $S$ is $a \cos \theta + h \sin \theta$; the length of the perpendicular from $C$ on the line of action of $W$ is $k \sin \theta$. Therefore

$$(Q+S)(a \cos \theta + h \sin \theta) - (P+S)(a \cos \theta - h \sin \theta) + Wk \sin \theta = 0;$$

therefore $$\tan \theta = \frac{(P-Q)a}{(P+Q+2S)h + Wk}.$$

This determines the position of equilibrium.

(1) When $P=Q$ we have $\tan \theta = 0$; thus the Balance is true: so that by making the arms equal and having the centre of gravity of the beam on the perpendicular from the fulcrum on the beam the first requisite is satisfied.

(2) For a given difference of $P$ and $Q$ the sensibility is obviously greater the greater $\tan \theta$ is; and for a given value of $\tan \theta$ the sensibility is greater the smaller the difference of $P$ and $Q$ is. Thus we may consider that the sensibility varies as $\tan \theta$ when $P-Q$ is constant; also that it varies inversely as $P-Q$ when $\tan \theta$ is constant; and so when both $\tan \theta$ and $P-Q$ vary the sensibility will be measured by $\frac{\tan \theta}{P-Q}$: see *Algebra for Beginners*, Art 389. Therefore the second requisite will be satisfied by making $(P+Q+2S)\frac{h}{a} + W\frac{k}{a}$ as small as possible.

(3) The stability is greater the greater the moment of the forces which tend to restore the equilibrium when it has been destroyed. Now this moment is

$$\{(P+Q+2S)h + Wk\} \sin \theta - (P-Q) a \cos \theta,$$

or supposing $P$ and $Q$ equal, it is

$$\{(P+Q+2S)h + Wk\} \sin \theta.$$

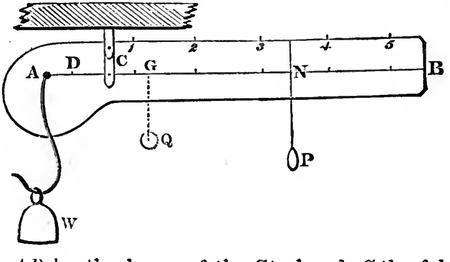
Hence to satisfy the third requisite this should be made as large as possible. This is, in part, at variance with the

second requisite. The two requisites may however both be satisfied by making $(P+Q+2S)\,h+Wk$ large, and $a$ large also; that is, by increasing the distances of the fulcrum from the beam and from the centre of gravity of the beam, and by lengthening the arms.

174. It may be observed, that the sensibility of a Balance is in general of more importance than the stability, since the eye can judge pretty accurately whether the beam makes equal oscillations on each side of the horizontal line; that is, whether the position of rest would be horizontal; if this be not the case, then the weights must be altered until the oscillations are nearly equal. Accordingly in practice the sensibility is secured at the expense of the stability; $k$ is made small; and $h$ very small. In fact $h$ is usually zero, or is extremely small; and thus the whole of this investigation might be simplified.

175. Another kind of Balance is that in which the arms are unequal, and the same weight is used to weigh different substances, by varying its distance from the fulcrum. The common Steelyard is of this kind.

176. *To graduate the common Steelyard.*




Let $AB$ be the beam of the Steelyard, $C$ the fulcrum. Let $A$ be the fixed point from which the substance to be weighed is suspended. Let $Q$ be the weight of the beam together with the hook or scale-pan at $A$; let $G$ be the centre of gravity of these. Let $P$ be a weight which can be placed at any distance from the fulcrum. Suppose that

the machine is in equilibrium with the beam horizontal when $P$ is suspended at $N$, and a substance of weight $W$ is suspended at $A$. Then, taking moments round $C$, we have by Art. 86,

$$W \,.\, AC - Q \,.\, CG - P \,.\, CN = 0;$$

therefore $$W = \frac{CN + \frac{Q}{P} CG}{AC} P.$$

On $GC$ produced through $C$, take the point $D$ such that

$$CD = \frac{Q}{P} CG \text{; then}$$

$$W = \frac{CN + CD}{AC} P = \frac{DN}{AC} P.$$

Now, let $DB$ be graduated by taking on it from $D$ distances equal respectively to $AC$, $2AC$, $3AC$, $4AC$,...: and let the figures 1, 2, 3, 4,... be placed over the points of graduation: these distances may also be subdivided if necessary. Then, by observing the graduation at $N$, we know the ratio of $W$ to $P$; and $P$ being a given weight, we know $W$.

In this manner any substance may be weighed.

177. The *sensibility* of the common Steelyard is greater the greater the distance between the two points at which $P$ must be suspended in order to balance two weights of given difference. Hence it will follow that the sensibility is increased by increasing $CA$, or by diminishing $P$. For suppose that $N'$ denotes the point of suspension of the moveable weight when the weight at $A$ is $W'$. Thus

$$P \,.\, DN' = W' \,.\, AC,$$

and $$P \,.\, DN = W \,.\, AC;$$

therefore $$P \,.\, NN' = (W' - W)\, AC;$$

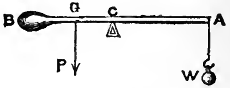
therefore $$NN' = \frac{(W' - W)\, AC}{P}.$$

Now $W' - W$ is supposed to be given; therefore $NN'$ varies as $\frac{AC}{P}$, and is increased by increasing $AC$, or by diminishing $P$.

Since the sensibility varies as $\frac{AC}{P}$ it is independent of the weight of the beam; it is also independent of the position of $N$, that is, a given Steelyard is equally sensible whatever be the weight which is to be determined.

178. Another kind of Balance is called the *Danish Steelyard.* This consists of a heavy beam which terminates in a knob at one end, and tho substance to be weighed is placed at the other end; the fulcrum is moveable.

179. *To graduate the Danish Steelyard.*




Let $AB$ be the beam; let $P$ be its weight, $G$ its centre of gravity.

Suppose that the machine is in equilibrium with the beam horizontal when the fulcrum is at $C$, and a substance of weight $W$ is suspended at $A$. Taking moments round $C$, we have, by Art. 86,

$$W.AC = P.CG = P(AG - AC);$$

therefore
$$AC = \frac{P.AG}{P+W}.$$

Hence, making $W = P, 2P, 3P, 4P, \ldots$ successively, we can mark on the beam the corresponding positions of the fulcrum. If intermediate graduations are required they must be determined by giving to $W$ intermediate values, as for example, $\frac{3}{2}P$, $\frac{8}{3}P$, $\frac{15}{4}P, \ldots$

It will be seen that if the successive values of $W$ form an Arithmetical Progression, the distances from $A$ of the corresponding graduations will form an Harmonical Progression.

## EXAMPLES. XII.

1. If a substance be weighed in a Balance having unequal arms, and in one scale appear to weigh $a$ lbs. and in the other scale $b$ lbs., shew that its true weight is $\sqrt{(ab)}$ lbs.

2. A body, the weight of which is one lb., when placed in one scale of a false Balance appears to weigh 14 ounces: find its weight when placed in the other scale.

3. The arms of a Balance are in the ratio of 19 to 20; the pan in which the weights are placed is suspended from the longer arm: find the real weight of a body which apparently weighs 38 lbs.

4. If a Balance be false, having its arms in the ratio of 15 to 16, find how much per lb. a customer really pays for tea which is sold to him from the longer arm at 3*s.* 9*d.* per lb.

The next six questions relate to the common Steelyard:

5. The moveable weight for which the Steelyard is constructed is one lb., and a tradesman substitutes a weight of two lbs., using the same graduations, thus giving his customers a weight which he says is the same number of times the moveable weight as before: shew that he defrauds his customers if the centre of gravity of the Steelyard be in the longer arm, and himself if it be in the shorter arm.

6. The moveable weight is one lb., and the weight of the beam is one lb.; the distance of the point of suspension from the body weighed is $2\frac{1}{2}$ inches, and the distance of the centre of gravity of the beam from the body weighed is 3 inches: find where the moveable weight must be placed when a body of 3 lbs. is weighed.

7. If the fulcrum divide the beam, supposed uniform, in the ratio of 3 to 1, and the weight of the beam be equal to the moveable weight, shew that the greatest weight which can be weighed is four times the moveable weight.

8. If the beam be uniform and its weight $\frac{1}{m}$ of the moveable weight, and the fulcrum be $\frac{1}{n}$ of the length of the

beam from one end, shew that the greatest weight which can be weighed is $\frac{2m(n-1)+n-2}{2m}$ times the moveable weight.

9. Find what effect is produced on the graduations by increasing the moveable weight.

10. Find what effect is produced on the graduations by increasing the density of the material of the beam.

11. A straight uniform Lever whose weight is 50 lbs. and length 6 feet, rests in equilibrium on a fulcrum when a weight of 10 lbs. is suspended from one extremity: find the position of the fulcrum and the pressure on it.

12. Two weights $P$ and $Q$ hang at the ends of a straight heavy Lever whose fulcrum is at the middle point: if the arms are both uniform, but not of the same weight, and the system be in equilibrium, shew that the difference between the weights of the arms equals twice the difference between $P$ and $Q$.

13. A uniform heavy rod $AB$, seven feet long, is supported in a horizontal position between two pegs $C$ and $D$, two feet apart, of which $C$ is half a foot from the end $A$: find the pressures on the pegs. If a force act upwards at a distance of half a foot from the end $B$, sufficient to remove all pressure from the peg $C$, shew that the pressure on the peg $D$ will be half of what it was before.

14. A bar of iron of uniform section and 12 feet long is supported by two men, one of whom is placed at one end: find where the other must be placed so that he may sustain three-fifths of the whole weight.

15. Two weights of 2 lbs. and 5 lbs. balance on a uniform heavy Lever, the arms being in the ratio of 2 to 1: find the weight of the Lever.

16. If a heavy uniform rod $c$ inches long be supported on two props at distances $a$ and $b$ inches from the ends, compare the pressures on the props.

17. A uniform heavy bar ten feet long and of given weight $W$ is laid over two props in the same horizontal line, so that one foot of its length projects over one of the

props. Find the distance between the props so that the pressure on the one may be double that on the other. Also find the pressures.

18. A straight Lever weighing 20 lbs. is moveable about a fulcrum at a distance from one extremity equal to one-fourth of its length: find what weight must be suspended from that extremity in order that the Lever may remain at rest in all positions.

19. A bent Lever is composed of two straight uniform rods of the same length, inclined to each other at $120^0$, and the fulcrum is at the point of intersection: if the weight of one rod be double that of the other, shew that the Lever will remain at rest with the lighter arm horizontal.

20. Two men carry a uniform beam 6 feet in length and weighing 2 cwt., and at 2 feet from one end a weight of 1 cwt. is placed: if one man have this end resting on his shoulder, find where the other man must support the beam in order that they may share the whole weight equally.

21. A cylindrical bar of lead a foot in length and 8 lbs. in weight is joined in the same straight line with a similar bar of iron 15 inches long and 6 lbs. in weight: find the point on which they will balance horizontally.

22. A uniform rod 10 feet long and 48 lbs. in weight is supported by a prop at one end: find the force which must act vertically upwards at a distance of 2 feet from the other end to keep the rod horizontal.

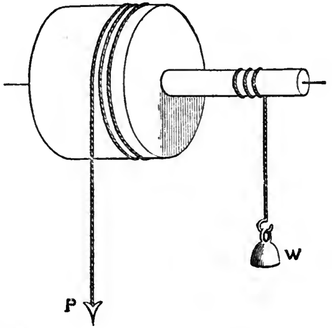
23. A straight uniform rod is suspended by one end: determine the position in which it will rest when acted on by a given horizontal force at the other end.

24. Two weights acting perpendicularly on a straight uniform Lever at its ends on opposite sides of the fulcrum balance: if one weight be double the other, and the weight of the Lever equal to their sum, find where the fulcrum must be.

25. If the common Steelyard be correctly constructed for a moveable weight $P$, shew that it may be made a correctly constructed instrument for a moveable weight $nP$ by suspending at the centre of gravity of the Steelyard a weight equal to $n-1$ times the weight of the Steelyard.

## XIII. *The Wheel and Axle. The Toothed Wheel.*

180. The present Chapter will be devoted to the Wheel and Axle, and the Toothed Wheel. It will be seen that these two Mechanical Powers are only modifications of the Lever.

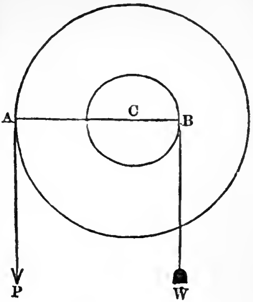



181. *The Wheel and Axle.* This machine consists of two cylinders which have a common axis; the larger cylinder is called the *Wheel*, and the smaller the *Axle.* The two cylinders are rigidly connected with the common axis, which is supported in a horizontal position so that the machine can turn round it. The Weight acts by a string which is fastened to the Axle and coiled round it; the Power acts by a string which is fastened to the Wheel and coiled round it. The Weight and the Power tend to turn the machine round the axis in opposite directions.

182. *When there is equilibrium on the Wheel and Axle, the Power is to the Weight as the radius of the Axle is to the radius of the Wheel.*

Let two circles having the common centre $C$ represent sections of the Wheel and Axle respectively, made by planes perpendicular to the axis of the cylinder.

It may be assumed, that the effects of the Power and the Weight will not be altered if we suppose them both to act in the same plane perpendicular to the axis. Let the string by which the power, $P$, acts leave the Wheel at $A$, and the string by which the weight, $W$, acts leave the Axle at $B$. Then $CA$ and $CB$ will be perpendicular to the lines of action of $P$ and $W$. We may regard




$ACB$ as a Lever of which $C$ is the fulcrum, and hence, by Art. 165, the necessary and sufficient condition for equilibrium is

$$\frac{P}{W} = \frac{CB}{CA}.$$

183. If we wish to take into account the thickness of the strings by which $P$ and $W$ act, we may consider that the line of action of each of these forces coincides with the middle of the respective strings. Thus, in the condition of equilibrium, $CA$ will denote the radius of the Wheel increased by half the thickness of the string by which $P$ acts, and $CB$ will denote the radius of the Axle increased by half the thickness of the string by which $W$ acts.

184. We have supposed that the Power in the Wheel and Axle acts by means of a string; but the Power may act by means of the hand, as in the familiar example of the machine used to draw up a bucket of water from a well.

A *windlass* and a *capstan* may also be considered as cases of the Wheel and Axle.

The windlass scarcely differs from the machine used to draw up water from a well: the windlass however has more than one fixed handle for the convenience of working it; or it may have a moveable handle which can be shifted from one place to another.

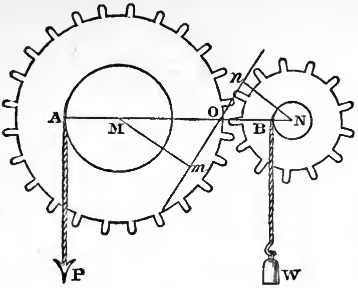
In the capstan the fixed axis of the machine is vertical; the hand which supplies the Power describes a circle in a horizontal plane, and the rope attached to what we call the Weight leaves the Axle in a horizontal direction.

185. In the Wheel and Axle, as described in Art. 182, the whole pressure on the fixed supports is equal to the *sum* of the Weight and the Power; for the machine resembles a Lever with parallel and like forces. If the Power be directed vertically upwards, the Power and the Weight being then on the same side of the axis of the machine, the whole pressure on the fixed support is equal to the *difference* of the Weight and the Power. In practice however with the windlass and the capstan, although $P$ and $W$ act at right angles to their respective radii, they do not necessarily act in parallel directions: in such cases the pressure on the fixed supports must be found by the Parallelogram of Forces. See Art. 168.

186. *The Toothed Wheel.* Let two circles of wood or metal have their circumferences cut into equal teeth at equal distances. Let the circles be moveable about axes perpendicular to their planes, and let them be placed with their axes parallel, so that their edges touch, one tooth of one circumference lying between two teeth of the other circumference. If one of the wheels of this pair be turned round its axis by any means, the other wheel will also be made to turn round its axis. Or a force which tends to turn one wheel round may be balanced by a suitable force which tends to turn the other wheel round.

187. *When there is equilibrium on a pair of Toothed Wheels, the moments of the Power and the Weight about the centres of their respective wheels are as the perpendiculars from the centres of the wheels on the direction of the pressure between the teeth in contact.*

Let $M$ and $N$ be the fixed centres of the wheels. Suppose the Power, $P$, and the Weight, $W$, to act by strings which are attached to axles concentric with the wheels. Let these strings leave the axles at $A$ and $B$ respectively. Then $MA$ and $NB$ will be perpendicular to the lines of action of $P$ and $W$.




Let $Q$ denote the mutual pressure at the point of contact of the teeth; so that a force $Q$ acts at the point of contact in opposite directions on the two wheels. Draw perpendiculars from $M$ and $N$ on the line of action of $Q$, meeting it at $m$ and $n$ respectively.

Then, since the wheel which can turn round $M$ is in equilibrium, the moments round $M$ must be equal; that is,

$$P \times AM = Q \times Mm.$$

Similarly, since the wheel which can turn round $N$ is in equilibrium,

$$W \times BN = Q \times Nn.$$

Therefore $$\frac{P \times AM}{W \times BN} = \frac{Q \times Mm}{Q \times Nn} = \frac{Mm}{Nn}:$$

this establishes the proposition.

Draw the straight line $MN$ meeting $mn$ at $O$: then, by similar triangles,

$$\frac{Mm}{Nn} = \frac{MO}{NO}.$$

If the teeth are very small compared with the radii of the wheels, $O$ will nearly coincide with the point of contact

of the teeth, and $MO$ and $NO$ will be nearly the radii of the wheels. Thus we have very nearly

$$\frac{\text{moment of } P \text{ round } M}{\text{moment of } W \text{ round } N} = \frac{\text{radius of Power-wheel}}{\text{radius of Weight-wheel}}.$$

188. In practice the machine is used to transmit motion; and then it is necessary to pay great attention to the form of the teeth, in order to secure uniform action in the machine, and to prevent the grinding away of the surfaces: on this subject however the student must consult works which treat specially of mechanism.

Toothed Wheels are extensively applied in all machinery, as in cranes and steam engines, and especially in watch-work and clock-work.

189. Wheels are sometimes turned by means of straps passing over their circumferences: in such cases the minute protuberances of the surfaces prevent the sliding of the straps. The strap passing partly round a wheel exerts a force on the wheel at both points where it leaves the wheel: the effect at each point would be measured by the moment of the tension of the strap round the centre of the wheel. If it were not for friction and the weight and stiffness of the strap the tension would be the same throughout; and so the action at one point of the wheel would balance the action at the other point. The subject of friction will be considered in Chapter XIX.

## EXAMPLES. XIII.

1. Find the radius of the Wheel to enable a Power of $1\frac{1}{2}$ lbs. to support a Weight of 28 lbs., the diameter of the Axle being 6 inches.

2. Find what Weight suspended from the Axle can be supported by 3 lbs. suspended from the Wheel, if the radius of the Axle is $1\frac{1}{2}$ feet, and the radius of the Wheel is $3\frac{1}{4}$ feet.

3. A man whose weight is 12 stone has to balance by his weight 15 cwt.: shew how to construct a Wheel and Axle which will enable him to do this.

4. A Weight of 14 ounces is supported by a certain Power on a Wheel and Axle, the radii being 28 inches and 16 inches respectively: if the radii were each shortened by 4 inches, find what Weight would be supported by the same Power.

5. If the radius of the Wheel be to the radius of the Axle as 8 is to 3, and two weights of 6 lbs. and 15 lbs. respectively be suspended from the circumferences of the Wheel and Axle, find which weight will descend. Supposing that the weight which tends to descend is supported by a prop, find the pressure on the prop and on the fixed supports of the Wheel and Axle.

6. The radius of the Axle of a capstan is 2 feet, and six men push each with a force of one cwt. on spokes 5 feet long: find the tension they will be able to produce in the rope which leaves the Axle.

7. The difference of the diameters of a Wheel and Axle is 2 feet 6 inches; and the Weight is equal to six times the Power: find the radii of the Wheel and the Axle.

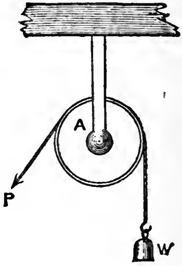
8. The radius of the Wheel being three times that of the Axle, and the string on the Wheel being only strong enough to support a tension of 36 lbs., find the greatest Weight which can be raised.

9. If the string to which the Weight is attached be coiled in the usual manner round the Axle, but the string by which the Power is applied be nailed to a point in the rim of the Wheel, find the position of equilibrium, the Power and the Weight being equal.

10. In the Wheel and Axle if the two ropes were coiled each on itself, and their thickness not neglected, find whether the ratio of the Power to the Weight would be increased or diminished as the Weight was raised, supposing the ropes of the same thickness.

## XIV. *The Pully.*

190. The *Pully* consists of a small circular plate or wheel which can turn round an axis passing through the centres of its faces, and having its ends supported by a framework which is called the *Block.* The circular plate has a groove cut in its edge to prevent a string from slipping off when it is put round the Pully.




191. Let $A$ denote a Pully the Block of which is fixed; and suppose a Weight attached to the end of a string passing round the Pully. If the string be pulled at the other end by a Power equal to the Weight there will be equilibrium.

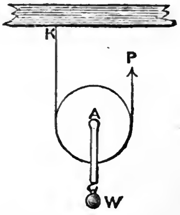
Thus a *fixed* Pully is an instrument by which we change the *direction* of a force without changing its *magnitude.* We have already adverted to this in Art. 28.

As we proceed with the present Chapter it will be seen that by the use of a *moveable* Pully we can gain mechanical advantage.

*Theoretically* the fact that the Pully can turn round its axis is not important; but *practically* it is very important. When the Pully can turn round it is found that the tension of the string is almost exactly the same on both sides of the Pully in the condition of equilibrium. But when the Pully cannot turn round it is found that there may be considerable difference between the tensions of the two parts of the string: this is owing to *Friction,* which we shall consider hereafter.

In all that follows we shall assume that the tension of a string is not changed when the string passes round a Pully. We shall always neglect the weight of the strings; and also the weight of the Pullies unless the contrary be stated.

192. *In a single moveable Pully with the strings parallel when there is equilibrium the Weight is twice the Power.*




Let a string pass round the Pully $A$, have one end fixed, as at $K$, and be pulled vertically upwards by a Power, $P$, at the other end.

Let a Weight, $W$, be attached to the Block of the Pully.

The tension of the string is the same throughout. Hence we may regard the Pully as acted on by two parallel forces, each equal to $P$, upwards, and by the force $W$ downwards. Therefore $W=2P$.

It may be observed that the line of action of $W$ must be equally distant from the two parts of the string; that is, it must pass through the centre of the Pully.

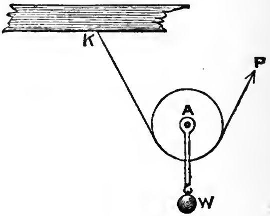
The pressure on the fixed point $K$ is equal to $P$, that is, to $\frac{1}{2}W$.

The string by which $P$ acts sometimes for convenience passes over a fixed Pully, and $P$ acts downwards.

193. The preceding Article will probably present no difficulty to the student; but perhaps the following remarks should be made. The Wheel and the Block of the Pully are really two distinct bodies; but when there is equilibrium we shall not disturb it by rigidly connecting the two bodies: thus we obtain one rigid body, and the condition of equilibrium follows by Art. 62. Sometimes the *principle of the Lever* is employed in obtaining this condition of equilibrium; the strict mode of employing the principle is as follows: The Wheel of the Pully is capable of turning round its axis, and for equilibrium the moments of the forces round this axis must be equal; this condition is satisfied if the axis be equidistant from the two parts of the string. The pressure on the axis is equal to the sum of the two forces; and this pressure is supported by the Block. Thus the Block is acted on by $2P$ upwards, and by $W$ downwards. Therefore $W=2P$

We may add that the action of the string in the preceding Article is best explained, for elementary purposes, by supposing all that part which is in contact with the Pully to become rigidly connected with it; thus we are left with a rigid body of Weight $W$ supported by the tensions of two strings acting at the points where the string leaves the Pully.

194. *To find the ratio of the Power to the Weight in the single moveable Pully with the parts of the string not parallel.*




Let a string pass round the Pully, $A$, have one end fixed, as at $K$, and be pulled by a Power, $P$, at the other end. Let a weight, $W$, be attached to the block of the Pully.

The tension of the string which passes round the Pully is the same throughout. Hence we may regard the Pully as acted on by two forces, each equal to $P$, and a force $W$. Therefore the line of action of $W$ must bisect the angle formed by the lines of action of the two forces $P$; that is, the two parts of the string must be equally inclined to the vertical. Suppose them each to make an angle $\alpha$ with the vertical. Then $W$ is equal and opposite to the resultant of two equal forces $P$, which are inclined at an angle $2\alpha$. Thus the ratio of $P$ to $W$ is known by the Parallelogram of Forces. By Art. 30, we have

$$W = 2P \cos \alpha.$$

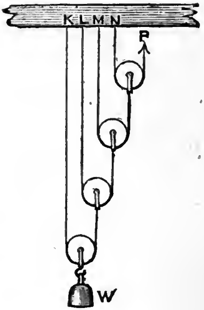
195. We now pass on to investigate the conditions of equilibrium of various combinations of Pullies.

196. *In the system of Pullies in which each Pully hangs by a separate string and all the strings are parallel, when there is equilibrium the Weight is equal to the Power multiplied by* $2^n$, *where n is the number of Pullies.*

In this system the string which passes round any Pully except the highest has one end attached to a fixed point,

and the other end attached to the block of the next higher Pully; the string which passes round the highest Pully has one end attached to a fixed point, and the other end supported by the Power.

Suppose there are four moveable Pullies. Let $W$ denote the weight, which is suspended from the block of the lowest Pully; and $P$ the Power which acts vertically upwards at the end of the string which passes under the highest Pully.




By the principle of the single moveable Pully, the tension of the string which passes under the lowest Pully is $\frac{W}{2}$; the tension of the string which passes under the next Pully is half of this, that is $\frac{W}{2^2}$; the tension of the string which passes under the next Pully is half of this, that is $\frac{W}{2^3}$; the tension of the string which passes under the next Pully is half of this, that is $\frac{W}{2^4}$. This last tension must be equal to the Power which acts at the end of the string. Therefore $P=\frac{W}{2^4}$, or $W=2^4P$.

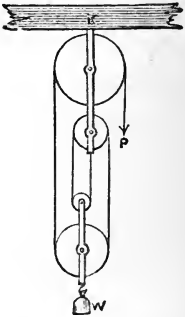
Similarly, if there be any number of moveable Pullies and $n$ denote this number, $W=2^nP$.

This system of Pullies is sometimes called the *First System* of Pullies.

197. Let $K, L, M, N$ denote the points at which the ends of the strings are fixed in the system of Pullies considered in the preceding Article. Then the pressure at $K$ is $\frac{W}{2}$, the pressure at $L$ is $\frac{W}{2^2}$, the pressure at $M$ is $\frac{W}{2^3}$, the pressure at $N$ is $\frac{W}{2^4}$. Hence the sum of these pres-

sures is $W\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}\right)$; by summing the Geometrical Progression, we find that this is $W\left(1-\frac{1}{2^4}\right)$. Thus the sum of these pressures together with the Power is equal to the whole Weight.

198. *In the system of Pullies in which the same string passes round all the Pullies and the parts of it between the Pullies are parallel, when there is equilibrium the Weight is equal to the Power multiplied by the number of parts of the string at the lower block.*




Suppose there are four parts of the string at the lower block.

Let $W$ denote the weight which is suspended from the lower block; and $P$ the power which acts vertically downwards at one end of the string. The tension of the string is the same throughout and is equal to $P$; thus we may regard the lower block as acted on by four parallel forces each equal to $P$ upwards, and by the force $W$ downwards. Therefore

$$W=4P.$$

Similarly, if there be any number of parts of the string at the lower block, and $n$ denote this number, $W=nP$.

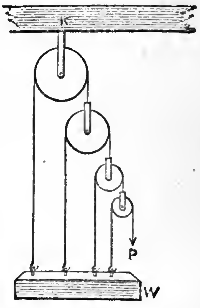
This system of Pullies is sometimes called the *Second System* of Pullies.

199. In the figure one end of the string is fastened to the upper block, and the number of parts of the string at the lower block is *even;* if one end of the string is fastened to the lower block the number of parts of the string at the lower block will be *odd.*

In the figure there are five parts of the string at the upper block, so that the pressure at the fixed point $K$ is $5P$, that is $W+P$.

200. *In the system of Pullies in which each string is attached to the Weight, and all the strings are parallel, when there is equilibrium the Weight is equal to the Power multiplied by* $2^n - 1$, *where* n *is the number of Pullies.*

In this system the string which passes round any Pully except the lowest has one end attached to the Weight, and the other end attached to the block of the next lower Pully; the string which passes round the lowest Pully has one end attached to the Weight, and the other end supported by the Power. The highest Pully is fixed; the others are moveable.




Suppose there are four Pullies. Let $W$ denote the Weight to which all the strings are attached; and $P$ the Power which acts vertically downwards at the end of the string which passes over the lowest Pully.

The tension of the string which passes over the lowest Pully is $P$; the tension of the string which passes over the next Pully is $2P$; the tension of the string which passes over the next Pully is twice this, that is $2^2P$; the tension of the string which passes over the next Pully is twice this, that is $2^3P$.

The Weight is equal to the sum of these tensions by Art. 111. Thus

$$W = P + 2P + 2^2P + 2^3P = P(1 + 2 + 2^2 + 2^3) = P(2^4 - 1).$$

Similarly, if there be any number of Pullies, and $n$ denote this number, $W = P(2^n - 1)$.

This system of Pullies is sometimes called the *Third System* of Pullies.

201. The pressure at the fixed point $K$ is $2 \times 2^3 P$, that is $2^4 P$, that is $W+P$.

202. We have hitherto neglected the weights of the Pullies; but it is easy to take account of them, and we shall now do so.

203. In Art. 192 let $w$ denote the weight of the Pully; we have only to put $W+w$ instead of $W$ in the condition of equilibrium. Thus $W+w=2P$.

Similarly, in Art. 194, $W+w=2P\cos\alpha$.

In Art. 198 let $w$ denote the whole weight of the Pullies at the lower block; then $W+w=4P$.

204. In Art. 196 let the weights of the four Pullies, beginning with the lowest, be $w$, $x$, $y$, $z$ respectively.

Then, the tension of the string which passes round the lowest Pully is $\frac{1}{2}(W+w)$; the tension of the next string is $\frac{1}{2^2}(W+w)+\frac{1}{2}x$; the tension of the next string is $\frac{1}{2^3}(W+w)+\frac{1}{2^2}x+\frac{1}{2}y$; the tension of the next string is $\frac{1}{2^4}(W+w)+\frac{1}{2^3}x+\frac{1}{2^2}y+\frac{1}{2}z$.

Thus $$P=\frac{W}{2^4}+\frac{w}{2^4}+\frac{x}{2^3}+\frac{y}{2^2}+\frac{z}{2}.$$

In fact, here $P$ supports simultaneously $W$ and $w$ by the aid of four Pullies, $x$ by the aid of three Pullies, $y$ by the aid of two Pullies, and $z$ by the aid of one Pully.

If the weight of each Pully is the same, and equal to $w$, we have

$$P=\frac{W}{2^4}+w\left(\frac{1}{2^4}+\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2}\right)=\frac{W}{2^4}+w\left(1-\frac{1}{2^4}\right).$$

Similarly, if $n$ denote the number of Pullies, and the weight of each Pully be $w$,

$$P=\frac{W}{2^n}+w\left(1-\frac{1}{2^n}\right), \text{ or } P-w=\frac{W-w}{2^n}.$$

205. In Art. 200 let $w$ denote the weight of the lowest Pully, $x$ that of the next, $y$ that of the next. Then the tension of the string which passes over the lowest Pully is $P$; the tension of the next string is $2P+w$; the tension of the next string is $2^2P+2w+x$; the tension of the next string is $2^3P+2^2w+2x+y$.

$$\begin{aligned}\text{Thus } W&=P(1+2+2^2+2^3)+w(1+2+2^2)+x(1+2)+y\\ &=P(2^4-1)+w(2^3-1)+x(2^2-1)+y.\end{aligned}$$

In fact here $W$ is supported by the simultaneous action of $P$ with the aid of four Pullies, $w$ with the aid of three, $x$ with the aid of two, and $y$ with the aid of one.

If the weight of each Pully is the same and equal to $w$, we have

$$W=P(2^4-1)+w(2^3+2^2+2-3)=P(2^4-1)+w(2^4-5).$$

Similarly, if $n$ denote the number of Pullies, and the weight of each Pully is $w$,

$$W=P(2^n-1)+w(2^n-n-1).$$

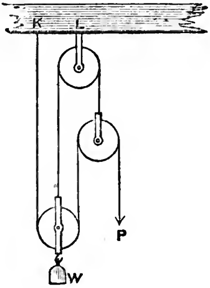
206. If we take the weights of the Pullies into account the expressions for the pressures on the fixed points which have been given in preceding Articles will require some alteration. There will be no difficulty in making these alterations; we will take Art. 197 as an example.

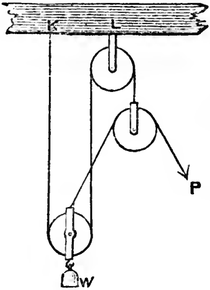
With the notation of Art. 204, the pressure at $K$ is $\frac{1}{2}(W+w)$; the pressure at $L$ is $\frac{1}{2^2}(W+w)+\frac{x}{2}$; the pressure at $M$ is $\frac{1}{2^3}(W+w)+\frac{x}{2^2}+\frac{y}{2}$; the pressure at $N$ is $\frac{1}{2^4}(W+w)+\frac{x}{2^3}+\frac{y}{2^2}+\frac{z}{2}$.

Hence the sum of these pressures

$$= W\left(1-\frac{1}{2^4}\right)+w\left(1-\frac{1}{2^4}\right)+x\left(1-\frac{1}{2^3}\right)+y\left(1-\frac{1}{2^2}\right)+z\left(1-\frac{1}{2}\right)$$
$$= W+w+x+y+z-P.$$

207. There are two systems of Pullies which are usually called Spanish Bartons: they will be understood from the annexed figures.




First, neglect the weights of the Pullies.

In the left-hand figure, $W=4P$. The pressure at the fixed point $K$ is $P$, and at the fixed point $L$ is $4P$.

In the right-hand figure, denoting by $2a$ the angle between the parts of the string round the upper moveable Pully, $W=5P\cos a$. The pressure at the fixed point $K$ is $2P\cos a$, and at the fixed point $L$ is $4P\cos a$.

Next, let $w$ denote the weight of the lower moveable Pully, and $x$ that of the upper moveable Pully.

In the left-hand figure $W+w=4P+x$. The pressure at the fixed point $K$ is $P$, and at the fixed point $L$ is $4P+2x$ together with the weight of the fixed Pully.

In the right-hand figure $W+w=5P\cos a+2x$. The pressure at the fixed point $K$ is $2P\cos a+x$, and at the fixed point $L$ is $4P\cos a+2x$ together with the weight of the fixed Pully.

208. It will be seen that in every system of Pullies we find the condition of equilibrium by beginning at one end of the system and determining in order the tensions of all the strings of the system. We have always begun with the *Power* end, except in Art. 196; and in that Article we might also have begun with the Power end.

## Examples. XIV.

1. In a single moveable Pully if the weight of the Pully be 2 lbs., find the force required to raise a Weight of 4 lbs.

2. If there be two strings at right angles to each other and a single moveable Pully, find the force which will support a Weight of $\sqrt{2}$ lbs.

3. A man stands in a scale attached to a moveable Pully, and a rope having one end fixed passes under the Pully, and then over a fixed Pully: find with what force the man must hold down the free end in order to support himself, the strings being parallel.

4. If on a Wheel and Axle the mechanical advantage be six times as great as on a single moveable Pully, compare the radii of the Wheel and the Axle.

The following ten Examples relate to the First System of Pullies; see Art. 196:

5. If $n=6$, and $P=28$ lbs., find $W$.

6. If $W=4$ lbs., and $P=1$ ounce, find $n$.

7. If $n=3$, find the consequence of adding one ounce to $P$, and ten ounces to $W$.

8. If a man support a Weight equal to his own, and there are three Pullies, find his pressure on the floor on which he stands.

9. If there are three Pullies, each weighing one lb., find the Power which will support a Weight of 17 lbs.

10. If there are three Pullies of equal weight, find the weight of each in order that a Weight of 56 lbs. attached to the lowest Pully may be supported by a Power of 7 lbs. 14 ounces.

11. If the weight of each Pully is $P$, find $W$ and the tension of each string.

12. If there are three Pullies each of weight $w$, and $W=P$, find $W$.

13. If there are two Pullies each of weight $4w$, and the Power be $3w$, shew that no Weight can be supported by the system.

14. In a system of three Pullies if a weight of 5 lbs. is attached to the lowest, 4 lbs. to the next, and 3 lbs. to the next, find the Power required for equilibrium.

The following six Examples relate to the Second System of Pullies; see Art. 198:

15. Find the number of parts of the string at the lower block in order that a Power of 4 ounces may support a Weight of 4 lbs.

16. Find the number of Pullies at the lower block if $P=12$ stone and $W=18$ cwt.

17. If there are four parts of the string at the lower block, find the consequence of adding one ounce to $P$, and three ounces to $W$.

18. If there are six parts of the string at the lower block, find the greatest Weight which a man weighing 10 stone can possibly raise.

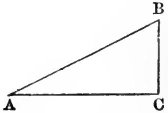
19. A man supports a Weight equal to half his own weight; if there are seven parts of the string at the lower block, find his pressure on the floor on which he stands.

20. Find what Weight can be supported if there are three Pullies at the lower block, the string being fastened to the upper block, and the weight of the lower block being three times the Power.

## XV. *The Inclined Plane.*

209. An *Inclined Plane* in Mechanics is a rigid plane inclined to the horizon.

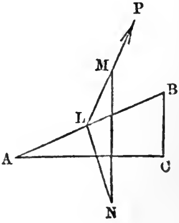
When an Inclined Plane is used as a Mechanical Power the straight lines in which the Power and the Weight act are supposed to lie in a vertical Plane perpendicular to the intersection of the Plane with the horizon. Thus the Inclined Plane is represented by a right-angled triangle, such as $ACB$; the horizontal side $AC$ is called the *base;* the vertical side $CB$ is called the *height;* and the hypotenuse $AB$ is called the *length.* The angle $BAC$ is the inclination of the Plane to the horizon.




The Plane is supposed perfectly *rigid*, and, unless the contrary be stated, it is supposed to be perfectly *smooth;* so that the Plane is assumed to be capable of supporting any amount of pressure which is exerted against it in a perpendicular direction.

210. *If a Weight be supported on an Inclined Plane at a point* L, *and* LM *be drawn in the direction of the Power, and* LN *at right angles to the Plane, so as to meet a vertical line at* M *and* N, *the Power is to the Weight as* LM *is to* MN.

Let $BAC$ be the Inclined Plane. Let a heavy body whose weight is $W$ be placed on it at any point $L$, and be kept at rest by a Power, $P$, acting in the direction $LM$. Let $MN$ be drawn vertically downwards, and $LN$ at right angles to the Plane.




The body at $L$ is acted on by three forces; the Power $P$ in the direction $LM$, its own Weight in a direction parallel to $MN$, and the resistance of the Plane in the direction $NL$.

Hence, by Art. 36, since there is equilibrium, the sides of the triangle $LMN$ are respectively proportional to the forces. Therefore

$$\frac{P}{W}=\frac{LM}{MN}.$$

Let $R$ denote the resistance of the Plane; then

$$\frac{R}{W}=\frac{NL}{MN}.$$

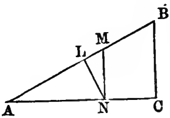
We may write these results thus:

$$P : W : R :: LM : MN : NL.$$

It is usual to consider separately two special cases of the general proposition; and this we shall do in the next two Articles.

211. *When there is equilibrium on the Inclined Plane, and the Power acts along the Plane, the Power is to the Weight as the height of the Plane is to the length.*

Let $W$ denote the Weight of a heavy body, and $P$ the Power. From any point $L$ in the Plane, draw $LN$ at right angles to the Plane, meeting the base at $N$; and draw $NM$ vertical, meeting the Plane at $M$.




Then the sides of the triangle $LMN$ are parallel to the directions of the forces which keep the heavy body at rest; therefore, by Art. 36,

$$\frac{P}{W}=\frac{LM}{MN}.$$

But the triangle $LMN$ is equiangular to the triangle $CBA$; for the angle $LMN$ is equal to the angle $ABC$, by Euclid, I. 29; the right angle $NLM$ is equal to the right angle $ACB$; and therefore the third angle $MNL$ is equal to the third angle $BAC$.

Hence, by Euclid, VI. 4,

$$\frac{LM}{MN}=\frac{CB}{BA};$$

therefore
$$\frac{P}{W}=\frac{CB}{BA}.$$

Let $R$ denote the resistance of the Plane; then

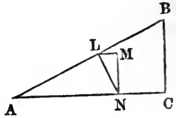
$$\frac{R}{W}=\frac{AC}{AB}.$$

We may write these results thus:

$$P : W : R :: CB : BA : AC.$$

212. *When there is equilibrium on the Inclined Plane, and the Power acts horizontally, the Power is to the Weight as the height of the Plane is to the base.*

Let $W$ denote the Weight of a heavy body, and $P$ the Power. From any point $L$ in the Plane draw $LN$ at right angles to the Plane, meeting the base at $N$; and draw $NM$ vertical meeting at $M$ the horizontal straight line drawn through $L$.




Then the sides of the triangle $LMN$ are parallel to the directions of the forces which keep the heavy body in equilibrium; therefore, by Art. 36,

$$\frac{P}{W}=\frac{LM}{MN}.$$

But the triangle $LMN$ is equiangular to the triangle $BAC$. For the angle $BLN$, being a right angle, is equal to the sum of the two angles $BAC$ and $ABC$; and $BLM$ is equal to $BAC$, by Euclid, I. 29: therefore $MLN$ is equal to $ABC$. And the right angles $LMN$ and $BCA$ are equal. Therefore the third angle $LNM$ is equal to the third angle $BAC$.

Hence, by Euclid, VI. 4,

$$\frac{LM}{MN} = \frac{BC}{CA};$$

therefore $$\frac{P}{W} = \frac{BC}{CA}.$$

Let $R$ denote the resistance of the Plane; then

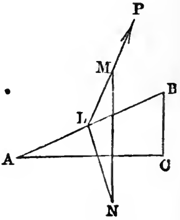
$$\frac{R}{W} = \frac{AB}{CA}.$$

We may write these results thus:

$$P : W : R :: BC : CA : AB.$$

213. We may obtain convenient expressions for the proportionate values of $P$, $W$, and $R$ by the aid of Trigonometry.

Let the angle $BAC = \alpha$, and the angle $MLB = \beta$; therefore the angle $MLN = 90^0 + \beta$.




Then

$$\begin{aligned} P : W : R &:: LM : MN : NL \\ &:: \sin LNM : \sin MLN : \sin NML \\ &:: \sin \alpha : \sin (90^0 + \beta) : \sin (90^0 - \alpha - \beta) \\ &:: \sin \alpha : \cos \beta : \cos (\alpha + \beta). \end{aligned}$$

214. In the figure of the preceding Article the resistance of the Plane is represented by the straight line $NL$; that is, the resistance acts from $N$ towards $L$. Thus if the body be placed *on* the Plane, and be in equilibrium, the straight line $LN$ must be *below* the Plane; that is, the sum of the angles $MLB$ and $BAC$ must be *less* than a right angle.

215. The results of Art. 213 may also be obtained by the method of resolving the forces given in Arts. 56, 57; and thus we obtain a good example of the method, and assistance in remembering the results.

Resolve the forces along the Plane: this gives

$$P \cos \beta = W \sin \alpha.$$

Resolve the forces at right angles to the Plane: this gives

$$P \sin \beta + R = W \cos \alpha.$$

Hence we deduce

$$R = W \cos \alpha - \frac{W \sin \alpha \sin \beta}{\cos \beta} = \frac{W \cos(\alpha + \beta)}{\cos \beta}.$$

The general formulæ of course include the particular cases of Arts. 211 and 212:

When the Power acts along the Plane $\beta = 0$; then

$$P = W \sin \alpha, \qquad R = W \cos \alpha.$$

When the Power acts horizontally $\beta = -\alpha$; then

$$P = W \tan \alpha, \qquad R = W \sec \alpha.$$

216. Perhaps it may seem that an *Inclined Plane* can scarcely be called a *Machine;* it is not obvious that it can be usefully employed like the other Mechanical Powers. But we may observe that if we have to raise a body we may draw it up an Inclined Plane by means of a Power which is less than the Weight of the body.

## EXAMPLES. XV.

In the following twelve Examples the Power is supposed to act along the Plane:

1. If the Weight be represented by the height of the Plane, shew what straight line represents the pressure on the Plane.

2. If $W = 12$ lbs., and the height of the Plane be to its base as 3 is to 4, find $P$.

3. If $W = 10$ lbs. and $P = 6$ lbs., find $R$.

4. If $P = R$, find the inclination of the Plane, and the ratio of $P$ to $W$.

5. If $P$ is to $R$ as 3 is to 4, express each of them in terms of $W$.

6. If $P=9$ lbs., find $W$ when the height of the Plane is 3 inches, and the base 4 inches.

7. An Inclined Plane rises 3 feet 6 inches for every 5 feet of length: if $W=200$, find $P$.

8. If the length of the Plane be 32 inches, and the height 8 inches, find the mechanical advantage.

9. When a certain Inclined Plane $ABC$, whose length is $AB$, is placed on $AC$ as base, a Power of 3 lbs. can support on it a Weight of 5 lbs.: find the Weight which the same Power could support if the Plane were placed on $BC$ as base, so that $AC$ is then the height of the Plane.

10. A railway train weighing 30 tons is drawn up an Inclined Plane of 1 foot in 60 by means of a rope and a stationary engine; find what number of lbs. at least the rope should be able to support.

11. A Weight of 20 lbs. is supported by a string fastened to a point in an Inclined Plane, and the string is only just strong enough to support a Weight of 10 lbs.: the inclination of the Plane to the horizon being gradually increased, find when the string will break.

12. If it takes twice the Power to support a given Weight on an Inclined Plane $ABC$ when placed on the side $AC$, that it does when the Plane is placed on the side $BC$, find the greatest Weight which a Power of one lb. can support on the Plane.

In the following four Examples the Power is supposed horizontal:

13. If $W=12$ lbs., and the base be to the length as 4 is to 5, find $P$.

14. If $W=48$ lbs., and the base be to the height as 24 is to 7, find $P$ and $R$.

15. If $R=2$ lbs., and $P=1$ lb., find $W$, and the inclination of the Plane.

16. If $W=12$ lbs., and $P=9$ lbs., find $R$.

17. If the Power which will support a Weight when acting along the Plane be half that which will do so acting horizontally, find the Inclination of the Plane.

18. If $R$ be the pressure on the Plane when the Power acts horizontally, and $S$ when it acts parallel to the Plane, shew that $RS = W^2$.

19. A Power $P$ acting along a Plane can support $W$, and acting horizontally can support $x$: shew that

$$P^2 = W^2 - x^2.$$

20. A Weight $W$ would be supported by a Power $P$ acting horizontally, or by a Power $Q$ acting parallel to the Plane: shew that

$$\frac{1}{Q^2} = \frac{1}{P^2} + \frac{1}{W^2}.$$

21. Give a geometrical construction for determining the direction in which the Power must act when it is equal to the Weight, but does not act vertically upwards; and shew that if $S$ be the pressure on the Plane in this case, and $R$ the pressure when the Power acts along the Plane, $S = 2R$.

22. The length of an Inclined Plane is 5 feet, and the height is 3 feet. Find into what two parts a Weight of 104 lbs. must be divided, so that one part hanging over the top of the Plane may balance the other part resting on the Plane.

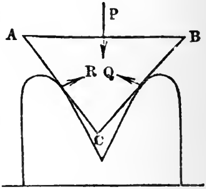
23. The inclination of a Plane is $30^0$; a particle is placed at the middle point of the Plane, and is kept at rest by a string passing through a groove in the Plane, and attached to the opposite extremity of the base: shew that the tension of the string is equal to the Weight of the particle.

24. A Weight of 20 lbs. is supported by a Power of 12 lbs. acting along the Plane: shew that if it were required to support the same Weight on the same Plane by a Power acting horizontally, the Power must be increased in the ratio of 5 to 4, while the pressure on the Plane will be increased in the ratio of 25 to 16.

## XVI. *The Wedge. The Screw.*

217. *The Wedge.* The Wedge is a solid body in the form of a *prism;* see Euclid, Book XI. Definitions. In the Wedge two parallel faces are equal and similar triangles, and there are three other faces which are rectangles.

The Wedge may be employed to separate bodies.




We may suppose the Wedge urged forward by a force $P$ acting on one of the rectangular faces, and urged backwards by two resistances $Q$ and $R$ on the other rectangular faces arising from the bodies which the Wedge is employed to separate. These forces may be supposed to act in one plane perpendicular to the rectangular faces; and we shall assume that the Wedge and the bodies are *smooth*, so that the force acting on each face is perpendicular to that face.

Let the triangle $ABC$ represent a section of the Wedge made by a plane perpendicular to its rectangular faces; and suppose the Wedge kept in equilibrium by the forces $P$, $Q$, $R$ perpendicular to $AB$, $BC$, $CA$ respectively: then by Art. 37

$$P : Q : R :: AB : BC : CA.$$

If $AC=BC$ the Wedge is called an *isosceles* Wedge; in this case $Q=R$, and $P : R :: AB : CA$.

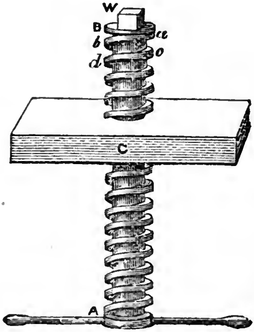
Let the angle $ACB$ be denoted by $2\alpha$, then when the Wedge is isosceles $AB=2AC \sin \alpha$, and

$$P : R :: 2 \sin \alpha : 1, \text{ so that } P=2R \sin \alpha.$$

218. There is very little value or interest in the preceding investigation, because the circumstances there supposed scarcely ever occur in practice. A nail is sometimes

given as an example of the Wedge, but when the nail is at rest the resistances on its sides are counterbalanced by friction and not by a Power on the head. The nail is indeed driven into its place by blows on the head; but it does not belong to Statics to investigate the effect of blows in producing motion.

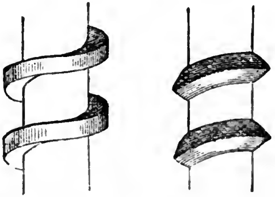
219. *The Screw.* The Screw consists of a right circular cylinder $AB$ with a uniform projecting thread $abcd$...traced round its surface, making a constant angle with straight lines parallel to the axis of the cylinder. This cylinder fits into a block $C$ pierced with an equal cylindrical aperture, on the inner surface of which is cut a groove the exact counterpart of the projecting thread $abcd$...



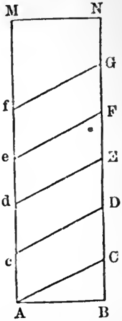

Thus when the block is fixed and the cylinder is introduced into it, the only manner in which the cylinder can move is backwards or forwards by turning round its axis.

220. In practice the forms of the threads of Screws may vary, as we see exemplified in the accompanying two figures.

The left-hand figure most nearly resembles that which is taken for investigation in elementary books; it is usual to disregard the thickness and the breadth of the projecting thread, that is to consider both of these as practically very small. We may form a good idea of the figure of the thread in the following geometrical manner:



Let $ABNM$ be any rectangle.




Take any point $C$ in $BN$, and make $CD$, $DE$, $EF$,... all equal to $BC$.

Join $CA$ and through $D$, $E$, $F$,... draw straight lines parallel to $CA$, meeting $AM$ at $c$, $d$, $e$,... Then if we conceive $ABNM$ to be formed into the convex surface of a right cylinder, the straight lines $AC$, $cD$, $dE$, $eF$,... will form the curve which determines the figure of the Screw.

Let the angle $CAB$ be denoted by $a$; then $CB = AB \tan a$; if $r$ be the radius of the right circular cylinder and $\pi$ express as usual the ratio of the circumference of a circle to its diameter, $AB = 2\pi r$; thus $CB = 2\pi r \tan a$. $CB$ is the distance between two consecutive threads of the Screw measured parallel to the axis. The angle $a$ may be called the *angle of the Screw.*

221. Suppose the axis of the cylinder to be vertical; and let a Weight $W$ be placed on the Screw. Then the Screw would descend unless prevented by some Power, $P$. This Power we shall suppose to act at the end of a horizontal arm firmly attached to the cylinder; the direction of the Power being horizontal and at right angles to the arm: the length between the axis of the cylinder and the point of application of the Power we shall call the *Power-arm.*

222. *When there is equilibrium on the Screw the Power is to the Weight as the distance between two adjacent threads is to the circumference of a circle having the Power-arm for radius.*

Let $r$ be the radius of the cylinder, $b$ the length of the Power-arm, $a$ the angle of the Screw. The Screw is acted on by the Weight $W$, the Power $P$, and resistances exerted by the block. These resistances act at various points of the block, but as the thread is supposed smooth they all act at right angles to the thread; thus their direc-

tions all make an angle $a$ with vertical straight lines. Denote these resistances by $R$, $S$, $T$,... Resolve each resistance into two components, one vertical and the other horizontal. Thus the vertical components are $R\cos a$, $S\cos a$, $T\cos a$,...; and the horizontal components are $R\sin a$, $S\sin a$, $T\sin a$,...

By reasoning as in Arts. 104 and 105 we find that there are two conditions which must hold when the machine is in equilibrium, namely:

The components parallel to the axis must balance each other, thus

$$W=(R+S+T+\ldots)\cos a;$$

and the moments of the forces round the axis must balance each other, thus

$$Pb=(R+S+T+\ldots)\,r\sin a.$$

Hence, by division,

$$\frac{Pb}{W}=\frac{r\sin a}{\cos a};$$

therefore $$\frac{P}{W}=\frac{r\sin a}{b\cos a}=\frac{2\pi r\tan a}{2\pi b}$$

$$=\frac{\text{distance between two consecutive threads}}{\text{circumference of circle of radius } b}.$$

223. The most common use of a Screw is not to support a Weight, but to exert a pressure. Thus suppose a fixed bar above the body denoted by $W$ in the figure of Art. 219; then, by turning the Screw, the body will be compressed between the head of the Screw and the fixed bar. A bookbinder's press is an example of this mode of using a Screw. The theory of the machine will be the same as in Art. 222; $W$ will now denote the pressure exerted parallel to the axis of the Screw by the body which is compressed.

## EXAMPLES. XVI.

1. A Wedge is right-angled and isosceles, and a force of 50 lbs. acts opposite to the right angle: determine the other two forces.

2. A Wedge is in the form of an equilateral triangle, and two of the forces are 40 lbs. each: find the third force.

3. Find the vertical angle of an isosceles Wedge when the pressure on the face opposite this angle is equal to half the sum of the two resistances.

4. The tangent of the angle of a Screw is $\frac{1}{4}$, the radius of the cylinder 4 inches, and the length of the Power-arm 2 feet: find the ratio of $W$ to $P$.

5. The circumference of the circle corresponding to the point of application of $P$ is 6 feet: find how many turns the Screw must make on a cylinder 2 feet long, in order that $W$ may be equal to 144 $P$.

6. The distance between two consecutive threads of a Screw is a quarter of an inch, and the length of the Power-arm is 5 feet: find what Weight will be sustained by a Power of 1 lb.

7. The angle of a Screw is $30^0$, and the length of the Power-arm is $n$ times the radius of the cylinder: find the mechanical advantage.

8. The length of the Power-arm is 15 inches: find the distance between two consecutive threads of the Screw, that the mechanical advantage may be 30.

9. A Screw having a head 12 inches in circumference is so formed that its head advances a quarter of an inch at each revolution: find what force must be applied at the circumference of the head that the Screw may produce a pressure of 96 lbs.

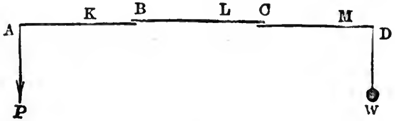
10. If a Screw makes $m$ turns in a cylinder a foot long, and the length of the Power-arm is $n$ feet, find the mechanical advantage.

## XVII. *Compound Machines.*

224. We have already spoken of the *mechanical advantage* of a machine, and have defined it to be the ratio of the Weight to the Power when the machine is in equilibrium; see Art. 166. Now we might *theoretically* obtain any amount of advantage by the use of any mechanical power. For example, in the Wheel and Axle the advantage is expressed by the ratio of the radius of the Wheel to the radius of the Axle; and this ratio can be made as great as we please: but *practically* if the radius of the Axle be too small the machine is not strong enough for use, and if the radius of the Wheel be too great the machine becomes of an inconvenient size.

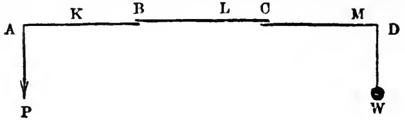
Hence it is found advisable to employ various compound machines, by which great mechanical advantage may be obtained combined with due strength and convenient size. We will now consider a few of these compound machines.

225. *Combination of Levers.*




Let $AB$, $BC$, $CD$ be three Levers, having fulcrums at $K$, $L$, $M$ respectively. Suppose all the Levers to be horizontal, and let the middle Lever have each end in contact with an end of one of the other Levers. Suppose the system in equilibrium with a Power, $P$, acting downwards at $A$, and a Weight, $W$, acting downwards at $D$.

Let $Q$ be the pressure at $B$ between the two Levers which are in contact there, and $R$ the pressure at $C$ between the two Levers which are in contact there; these pressures may be supposed to act vertically.




Since the Lever $AKB$ is in equilibrium $\frac{Q}{P} = \frac{AK}{BK}$;

since the Lever $BLC$ is in equilibrium $\frac{R}{Q} = \frac{BL}{CL}$;

and since the Lever $CMD$ is in equilibrium $\frac{W}{R} = \frac{CM}{DM}$.

Hence, by multiplication, $\frac{W}{P} = \frac{AK}{BK} \times \frac{BL}{CL} \times \frac{CM}{DM}$.

Hence the mechanical advantage of the combination of Levers is equal to the product of the mechanical advantages of the component Levers.

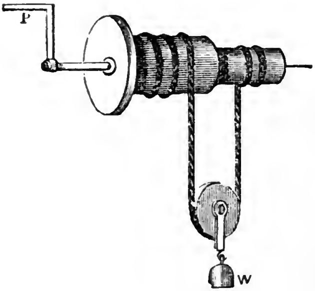
The result holds even if $Q$ and $R$ do not act vertically. Suppose for example that $Q$ does not act vertically, but in some other direction; let $k$ and $l$ denote the lengths of the perpendiculars from $K$ and $L$ on this direction. Then we have $\frac{Q}{P} = \frac{AK}{k}$, $\frac{R}{Q} = \frac{l}{CL}$. But by similar triangles $\frac{l}{k} = \frac{BL}{BK}$; and thus the value of $\frac{W}{P}$ is the same as before.

226. Combinations of Wheels and Axles are frequently used. The Wheel of one component is made to act on the Axle of the next component by means of teeth, or of a strap. It may be shewn that the mechanical advantage of

the combination is the product of the mechanical advantages of the components; that is, we shall have

$$\frac{W}{P} = \frac{\text{Product of the radii of the Wheels}}{\text{Product of the radii of the Axles}}.$$

227. *The Differential Axle, or Chinese Wheel.*




This machine may be considered as a combination of the Wheel and Axle with a single moveable Pully.

Two cylinders of unequal radii have a common axis with which they are rigidly connected; the axis is supported in a horizontal position so that the machine can turn round. A string has one end fastened to the larger cylinder, is coiled several times round this cylinder, then leaves it, passes under a moveable Pully, and is coiled round the smaller cylinder, to which the other end is fastened. The string is coiled in opposite ways round the two cylinders, so that when it winds off one it winds on the other. A weight $W$ is suspended from the moveable Pully. The equilibrium is maintained by a Power, $P$, applied at the end of a handle attached to the axis.

Let $a$ denote the radius of the larger cylinder, $b$ the radius of the smaller cylinder, $c$ the length of the arm at which the Power acts.

Suppose the machine in equilibrium, and the parts of the string on both sides of the Pully to be vertical.

The tension of the string is the same throughout; and is equal to $\frac{1}{2}$ $W$ by Art. 192.

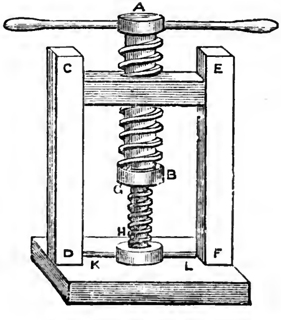
At the point where the string leaves the larger cylinder the tension tends to turn the machine round in one direction, and at the point where the string leaves the smaller cylinder the tension tends to turn the machine round in the opposite direction. Hence, taking moments round the axis, we have by Art. 100,

$$Pc + \frac{1}{2}\,Wb = \frac{1}{2}\,Wa;$$

therefore
$$\frac{W}{P} = \frac{2c}{a-b}.$$

Thus we see that by taking $a$ and $b$ very nearly equal we can obtain any amount of mechanical advantage.

228. *Hunter's Screw, or the Differential Screw.*




$AB$ is a right circular cylinder, having a Screw traced on its surface; this fits into a corresponding groove cut in the block $CE$, which forms part of the rigid framework $CDFE$.

$AB$ is hollow, and has a thread cut in its inner surface, so that a second Screw $GH$ can work in it. The second Screw does not turn round, for it has a cross bar $KL$ the ends of which are constrained by smooth grooves, so that the piece $GHKL$ can only move up and down. The machine is used to produce a great pressure on any substance placed between $KL$ and the fixed base on which the framework $CDFE$ stands.

Let $P$ denote the Power applied by a handle at the top of the outer Screw. Let $W$ denote the pressure exerted below $KL$ in the state of equilibrium.

Let $a$ denote the angle of the outer Screw, $r$ the radius of the cylinder; let $a'$ denote the angle of the inner Screw, $r'$ the radius of the cylinder. Let $b$ be the length of the arm at which the Power acts. We shall now proceed as in Art. 222.

Let the resistances which act between the outer Screw and the block be denoted by $R, S, T,...$; and those between the two Screws by $R', S', T',...$

Then, as in Art. 222, since the outer Screw is in equilibrium,

$$(R' + S' + T' + ...) \cos a' = (R + S + T + ...) \cos a,$$

$$Pb = (R + S + T + ...) r \sin a - (R' + S' + T' + ...) r' \sin a';$$

and since the inner Screw is in equilibrium,

$$W = (R' + S' + T' + ...) \cos a'.$$

From the first and third of these equations

$$W = (R + S + T + ...) \cos a;$$

and then from the second equation

$$Pb = W(r \tan a - r' \tan a');$$

therefore $$\frac{W}{P} = \frac{b}{r \tan a - r' \tan a'}.$$

Thus we see that by making $r \tan a$ and $r' \tan a'$ very nearly equal we can obtain any amount of mechanical advantage.

229. We see by what has been said respecting the mechanical powers and compound machines that we can obtain any amount of mechanical advantage. But it must be observed that we have hitherto considered machines in the state of equilibrium, that is as used to *support* weights; practically however machines are more commonly used to *move* weights. Now it is found that although with the aid of a machine we can move a Weight by a Power much smaller than the Weight, yet in order to make the Weight

move through a line of any length the Power must describe a much longer line. Let us take as a simple example the single moveable Pully described in Art. 192; suppose the Power somewhat greater than half the Weight, so that instead of equilibrium we have the Power prevailing over the Weight. If we have to raise the Weight through one foot, the vertical part of the string which ends at the fixed point $K$ must be shortened one foot, and this requires the end at which the Power $P$ acts to move through two feet, in order to keep the string stretched. Thus the length of the line which $P$ describes is to the length of the line which $W$ describes as $W$ is to $P$.

The principle is popularly enunciated thus: *what is gained in power is lost in speed.* We will give another illustration of it.

We will take the case of the *Differential Axle;* see Art. 227. Suppose the cylinders to turn once completely round so as to raise the Weight; then the point of application of the Power $P$ moves round the circumference of a circle of radius $c$, that is describes a length $2\pi c$. The depth of the centre of the Pully below the axis is half the sum of the lengths of the two parts of the string. Now in turning once round the length $2\pi a$ is wound on one cylinder, and the length $2\pi b$ is wound off the other. Thus the Weight is raised through the height $\frac{1}{2}(2\pi a - 2\pi b)$, that is through $\pi(a-b)$. Therefore

$$\frac{\text{length described by } P}{\text{length described by } W} = \frac{2\pi c}{\pi(a-b)} = \frac{2c}{a-b} = \frac{W}{P}.$$

## EXAMPLES. XVII.

1. Three horizontal Levers $AKB$, $BLC$, $CMD$ without weight, whose fulcrums are $K$, $L$, $M$, act on one another at $B$ and $C$ respectively, and are kept in equilibrium by Weights of 1 lb. at $A$ and 24 lbs. at $D$: if $AK$, $KB$, $BC$, $CM$, $MD$ are equal to 2, 1, 4, 4, 1 feet respectively, find the position of $L$ and the pressure on it.

2. In a combination of Wheels and Axles each of the radii of the Wheels is five times the radius of the corresponding Axle: if there be three Wheels and Axles determine what Power will balance a Weight of 375 lbs.

3. A rope, the ends of which are held by two men $A$ and $B$, passes over a fixed Pully $L$, under a moveable Pully $M$, and over another fixed Pully $N$. A Weight of 120 lbs. is suspended from $M$. Supposing the different parts of the rope to be parallel find with what force $A$ and $B$ must pull to support the Weight.

4. In the preceding Example if $B$ fastens his end of the rope to the Weight find whether any change takes place in the force which $A$ must exert.

5. $A$ is a fixed Pully; $B$ and $C$ are moveable Pullies, A string is put over $A$; one end of it passes under $C$ and is fastened to the centre of $B$; the other end passes under $B$ and is fastened to the centre of $A$. Compare the Weights of $B$ and $C$ that the system may be in equilibrium, the strings being parallel.

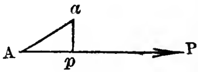
6. Two unequal Weights connected by a fine string are placed on two smooth Inclined Planes which have a common height, the string passing over the intersection of the Planes: find the ratio between the Weights when there is equilibrium.

7. A Weight $W$ is supported on an Inclined Plane by a string along the Plane. The string passes over a fixed Pully, and then under a moveable Pully to which a weight $W$ is attached, and having the parts of the string on each side of it parallel; the end of the string is attached to a fixed point: shew that in order that the system may be in equilibrium the height of the Plane must be half its length.

8. In an Inclined Plane if the Power $P$ be the tension of a fine string which passes along the Plane and over a small fixed Pully and is attached to a Weight hanging freely, shew that if $P$ be pulled down through a given length the height of the centre of gravity of $P$ and $W$ will remain unchanged.

## XVIII. *Virtual Velocities.*

230. We have already drawn attention to a very remarkable fact with respect to machines, which is popularly expressed in these words: *what is gained in power is lost in speed.* This fact is included in the general *Principle of Virtual Velocities*, which we will now consider.



231. Suppose that $A$ is the point of application of a force $P$; conceive the point $A$ to be moved in any direction to a new position $a$ at a very slight distance, and from $a$ draw a perpendicular $ap$ on the line of action of the force $P$: then $Ap$ is called *the virtual velocity of the point A with respect to the force P;* and the complete phrase is abbreviated, sometimes into *the virtual velocity of the point A*, and sometimes into *the virtual velocity of the force P.*

The virtual velocity is considered to be positive or negative according as $p$ falls on the direction of $P$ or on the opposite direction. Thus in the figure the virtual velocity is positive.

We see that $Ap = Aa \cos aAp$.

232. Now it is found that the following remarkable proposition is true: *suppose a system of forces in equilibrium, and imagine the points of application of the forces to undergo very slight displacements, then the algebraical sum of the products of each force into its virtual velocity vanishes; and conversely if this sum vanishes for all possible displacements the system of forces is in equilibrium.*

This proposition is called the *Principle of Virtual Velocities.*

233. We shall not attempt to demonstrate this important principle, or even to explain it fully; the student

may hereafter consult the larger work on Statics. We may however notice two points.

The displacements which the principle contemplates are such as do not destroy the connexion of the points of application of the forces with each other. Thus any rigid body must be conceived to be moved as a whole, without separation into parts; also any rods or strings which transmit forces must be conceived to remain unbroken.

The word *virtual* is used to intimate that the displacements are not really made but only *supposed* to be made. The word *velocities* is used because we may conceive all the points of application of the forces to move into their new positions in the *same time*, and then the lengths of the paths described are proportional to the velocities in the ordinary meaning of this word. But there is no necessity for introducing this conception, and it would probably be advantageous for beginners if the term *virtual velocity* could be changed into *virtual displacement.*

234. In the present work we shall follow the usual course of elementary writers, and shew that the Principle of Virtual Velocities holds for all the Mechanical Powers, by special examination of each case.

Thus in every case which we shall examine there will be two forces, the Power, $P$, and the Weight, $W$; and we shall have to establish the result

$$P \times \text{its virtual velocity} + W \times \text{its virtual velocity} = 0.$$

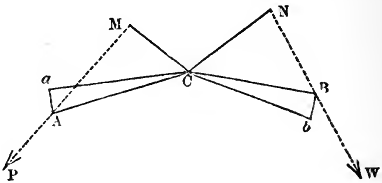
We shall find that in every case the virtual velocities of $P$ and $W$ will have *opposite* signs; but as there are only two forces we shall not fall into any confusion by dropping the distinction between positive and negative virtual velocities. We shall accordingly shew that in every case we have *numerically*

$$P \times \text{its virtual velocity} = W \times \text{its virtual velocity}.$$

Suppose for example that in any case of equilibrium $P$ is one-tenth of $W$; then the virtual velocity of $P$ must be ten times that of $W$. Thus the reader will see after carefully studying this Chapter that the Principle of Virtual Velocities includes the fact stated in Art. 230.

235. *To demonstrate the Principle for the Lever.*

Let $ACB$ be a Lever, having its fulcrum at $C$; and kept in equilibrium by forces $P$ and $W$ which act at $A$ and $B$ respectively.




Suppose the Lever to be turned round $C$ so as to come into the position $aCb$. Join $Aa$ and $Bb$.

The angle $ACa$ = the angle $BCb$; denote it by $2\theta$.

Let $CM$ and $CN$ be perpendiculars from $C$ on the lines of action of $P$ and $W$ respectively. Let the angle $MAC = \alpha$, and the angle $NBC = \beta$.

Then $\qquad CAa = 90^0 - \theta$, and $CBb = 90^0 - \theta$.

The displacement of $A$ resolved along $AM$

$$= Aa \cos MAa = Aa \cos (90^0 - \alpha - \theta) = Aa \sin (\alpha + \theta).$$

The displacement of $B$ resolved along $NB$

$$= Bb \cos (180^0 - bBC - CBN)$$

$$= Bb \cos (90^0 - \beta + \theta) = Bb \sin (\beta - \theta).$$

Therefore $\dfrac{\text{Resolved displacement of } A}{\text{Resolved displacement of } B}$

$$= \frac{Aa \sin (\alpha + \theta)}{Bb \sin (\beta - \theta)} = \frac{CA}{CB} \frac{\sin (\alpha + \theta)}{\sin (\beta - \theta)},$$

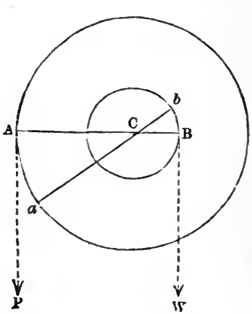
for the triangle $ACa$ is similar to the triangle $BCb$.

Now when $\theta$ is made indefinitely small the right-hand side of this equation becomes $\dfrac{CA \sin \alpha}{CB \sin \beta}$ or $\dfrac{CM}{CN}$, which is

equal to $\frac{W}{P}$ by the principle of the Lever.

Hence ultimately, $P$ multiplied by the resolved displacement of its point of application is equal to $W$ multiplied by the resolved displacement of its point of application.

236. *To demonstrate the Principle for the Wheel and Axle.*




Let two circles having the common centre $C$ represent sections of the Wheel and Axle respectively.

Let the machine be in equilibrium with the Power $P$ acting downwards at $A$, and the Weight $W$ acting downwards at $B$.

Suppose the machine to be turned round its axis so that $A$ comes to $a$, and $B$ comes to $b$; then $aCb$ is a straight line.

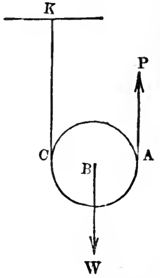
The displacement of $A$ resolved along the line of action of $P$ is $Ca \sin ACa$; the displacement of $B$ resolved along the line of action of $W$ is $Cb \sin BCb$.

$$\text{Therefore } \frac{\text{Resolved displacement of } A}{\text{Resolved displacement of } B}$$

$$= \frac{Ca \sin ACa}{Cb \sin BCb} = \frac{Ca}{Cb} = \frac{CA}{CB} = \frac{W}{P}.$$

Hence $P$ multiplied by the resolved displacement of its point of application is equal to $W$ multiplied by the resolved displacement of its point of application.

237. *To demonstrate the Principle for the single moveable Pully with parallel strings.*




Suppose the Weight to be raised through any height $s$; then the part $KC$ of the string between the fixed end and the Pully must be shortened by $s$: and to keep the string stretched the end at which $P$ acts must be raised through the height $2s$. Therefore the point of the string which is on the line of action of $P$, and in contact with the Pully, is raised through the height $2s$. Denote this point of the string, at which $P$ may be supposed to act, by $A$; and denote the centre of the Pully, at which $W$ may be supposed to act, by $B$. Then

$$\frac{\text{Displacement of } A}{\text{Displacement of } B} = \frac{2s}{s} = \frac{2}{1} = \frac{W}{P}.$$

Hence $P$ multiplied by the displacement of its point of application is equal to $W$ multiplied by the displacement of its point of application.

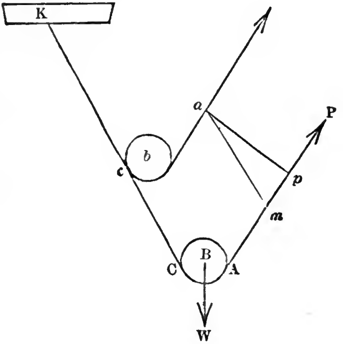
238. *To demonstrate the Principle for the single moveable Pully with strings not parallel.*

Let the system be displaced so that the strings are still *inclined at the same angle as before*, the part of the string with the fixed end being kept in its original direction. Let $2a$ be the angle between the parts of the string.

Let $A$ denote the point of the string where the string leaves the Pully, at which we may suppose the Power to act. Let $A$ be displaced to $a$; draw $ap$ perpendicular to the line of action of $P$: then $Ap$ is the resolved displacement of $A$.

Let $B$, the centre of the Pully, be displaced to $b$: then $Bb \cos a$ is the displacement of $B$ resolved in the direction of the Weight. Draw $am$ parallel to $Bb$ meeting the line

of action of $P$ at $m$. Let $C$ and $c$ denote the points at which the part of the string with the fixed end $K$ leaves the Pully in its two positions.




Now $\qquad Ap = Am + mp = Am + am \cos 2a.$

But since the length of the part $KCA$ of the string is equal to the length of the part $Kca$ we have $Am = Cc = Bb$.

Also it may be shewn that $am$ is equal to $Bb$.

Thus $\qquad Ap = Bb\,(1 + \cos 2a) = 2Bb \cos^2 a.$

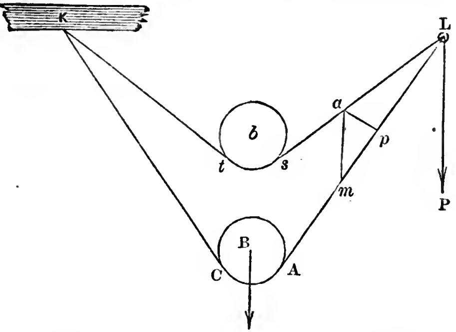
Therefore

$$\frac{\text{Resolved displacement of } A}{\text{Resolved displacement of } B} = \frac{2Bb \cos^2 a}{Bb \cos a} = 2 \cos a = \frac{W}{P}.$$

Hence $P$ multiplied by the resolved displacement of its point of application is equal to $W$ multiplied by the resolved displacement of its point of application.

239. In the preceding Article we considered such a displacement as left the two parts of the string inclined at the same angle as they were originally: it is however usual to consider another displacement, in which this condition is not fulfilled. We will now give the investigation; but it involves so many approximations, instead of exact equalities, that it is difficult for a beginner.

Let $K$ be the fixed end of the string. Suppose that part of the string to which the Power is applied to pass




over a small fixed peg or Pully at $L$, such that $K$ and $L$ are in the same horizontal line. Let $2a$ be the angle between the parts of the string. Let the system be displaced so that the point of application of the Weight rises vertically through the height $Bb$. Let $A$ denote the point of the string where the string leaves the Pully, at which we may suppose the Power to act; and suppose $A$ displaced to $a$. Draw $ap$ perpendicular to $AL$; then $Ap$ is the resolved displacement of $A$.

Let $AC$ be the part of the string in contact with the Pully in the original position, and $st$ the part of the string in contact with the Pully in the second position.

Draw $am$ vertical meeting $AL$ at $m$: then the angle $amp = a$, and $mp = am \cos a$.

Now when the displacement is very small we may consider that $am = Bb$. For if $sa$ were parallel to $Am$ then $am$ would be parallel and equal to $Bb$; and since the angle between $sa$ and $Am$ is supposed very small we may treat these straight lines as if they were parallel.

Also when the displacement is very small we may consider $Am = Bb \cos a$. For the length of the part $KCA$ of the string is exactly equal to the length of the part $Kta$; the parts in contact with the Pully, $AC$ and $st$, will be

very nearly equal: and therefore we may consider that $as = KC - Kt$. Now if $KC$ and $Kt$ coincided in direction we should have $KC - Kt = Bb \cos a$.

And we regard $Am$ as parallel and equal to $sa$, so that we take $Am = as = Bb \cos a$.

Thus $Ap = Am + mp = 2Bb \cos a$.

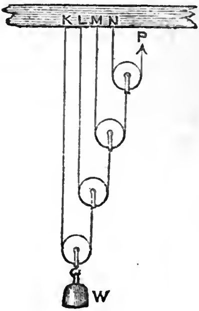
Therefore, when the displacement is very small,

$$\frac{\text{Resolved displacement of } A}{\text{Resolved displacement of } B} \quad \frac{2Bb \cos a}{Bb} = 2 \cos a = \frac{W}{P}.$$

Hence $P$ multiplied by the resolved displacement of its point of application is equal to $W$ multiplied by the resolved displacement of its point of application.

240. *To demonstrate the Principle for the First System of Pullies.*

Let there be four Pullies. Suppose the Weight raised through a height $s$. Then the lowest Pully is raised through a height $s$, the next Pully is raised through a height $2s$, the next Pully is raised through a height $4s$, and the highest Pully is raised through a height $8s$. And as the highest Pully is raised through a height $8s$ the point at which the Power acts is raised through a height $16s$: see Art. 237.



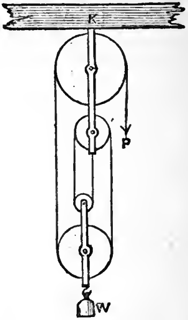

Therefore

$$\frac{\text{Displacement of the point at which } P \text{ acts}}{\text{Displacement of the point at which } W \text{ acts}} = \frac{16s}{s} = \frac{2^4}{1} = \frac{W}{P}.$$

Hence $P$ multiplied by the displacement of its point of application is equal to $W$ multiplied by the displacement of its point of application.

In a similar manner the result may be established whatever be the number of Pullies.

241. *To demonstrate the Principle for the Second System of Pullies.*




Let there be four parts of the string at the lower block.

Suppose the Weight raised through a height $s$, then each of the four parts of the string at the lower block is shortened by $s$; and therefore the point at which the Power acts must descend through $4s$.

Therefore

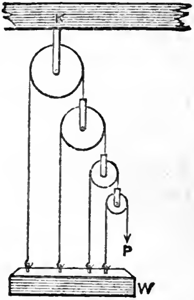
$$\frac{\text{Displacement of the point at which } P \text{ acts}}{\text{Displacement of the point at which } W \text{ acts}} = \frac{4s}{s} = \frac{4}{1} = \frac{W}{P}.$$

Hence $P$ multiplied by the displacement of its point of application is equal to $W$ multiplied by the displacement of its point of application.

In a similar manner the result may be established whatever be the number of parts of the string at the lower block.

242. In this system of Pullies it is easily seen that when the Weight is raised through a height $s$ a length $s$ of string passes round the highest Pully of the lower block, a length $2s$ passes round the lowest Pully of the upper block, a length $3s$ passes round the next Pully of the lower block, a length $4s$ passes round the next Pully of the upper block; and so on if there are more Pullies. Hence it follows that if the radii of the Pullies at the lower block are proportional to 1, 3, 5,... the Pullies will turn through equal angles in equal times when the machine is used to raise a Weight. Thus all these Pullies may be connected together so as to turn on one common axis. Also if the radii of the Pullies at the upper block are proportional to 2, 4, 6,... these Pullies may be connected so as to turn on one common axis. This arrangement was invented by James White, and is called *White's Pully*.

243. *To demonstrate the Principle for the Third System of Pullies.*




Let there be four Pullies. Suppose the Weight raised through a height $s$. Then the highest moveable Pully descends through a depth $s$. The next Pully descends through $2s$ in consequence of the descent of the Pully above it, and through $s$ besides in consequence of the ascent of the Weight; thus it descends through $(2+1)s$ on the whole. The next Pully descends through twice this depth in consequence of the descent of the Pully immediately above it, and through $s$ besides in consequence of the ascent of the Weight; thus it descends through $2(2+1)s+s$ on the whole, that is through $(2^2+2+1)s$. Similarly the end of the string at which the Power acts descends through $(2^3+2^2+2+1)s$.

Therefore $\dfrac{\text{Displacement of the point at which } P \text{ acts}}{\text{Displacement of the point at which } W \text{ acts}}$

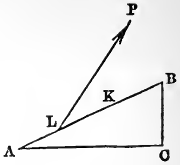
$$=\frac{(2^3+2^2+2+1)s}{s}=\frac{2^3+2^2+2+1}{1}=\frac{W}{P}.$$

Hence $P$ multiplied by the displacement of its point of application is equal to $W$ multiplied by the displacement of its point of application.

In a similar manner the result may be established whatever be the number of Pullies.

244. *To demonstrate the Principle for the Inclined Plane.*

Let a Weight $W$ be supported at $L$ on an Inclined Plane by a Power, $P$, the direction of which makes an




angle $\beta$ with the Plane. Suppose the Weight displaced along the Plane to a point $K$. Then the displacement of $L$ resolved along the direction of $P$ is $LK \cos \beta$; and the displacement of $L$ resolved along the direction of $W$ is $LK \sin \alpha$: see Art. 231.

Therefore

$$\frac{\text{Resolved displacement of the point at which } P \text{ acts}}{\text{Resolved displacement of the point at which } W \text{ acts}}$$

$$= \frac{LK \cos \beta}{LK \sin \alpha} = \frac{\cos \beta}{\sin \alpha} = \frac{W}{P}.$$

Hence $P$ multiplied by the resolved displacement of its point of application is equal to $W$ multiplied by the resolved displacement of its point of application.

245. *To demonstrate the Principle for the Screw.*

Suppose the Power-arm to make a complete revolution, then the Weight would rise through a height equal to the distance between two consecutive threads measured parallel to the axis of the Screw. The path of the end of the Power-arm, estimated on the horizontal plane in which the Power is supposed to act originally, is the circumference of a circle having the Power-arm for radius. If the Power-arm instead of making a complete revolution makes only a small part of a revolution, the *ratio* between the displacements of the end of the Power-arm and of the Weight remains the same. Therefore

$$\frac{\text{Resolved displacement of the point of application of } P}{\text{Displacement of } W}$$

$$= \frac{\text{circumference of circle with Power-arm for radius}}{\text{distance between two consecutive threads}} = \frac{W}{P}.$$

Hence $P$ multiplied by the resolved displacement of its point of application is equal to $W$ multiplied by the displacement of its point of application.

246. The student will not find any difficulty in shewing that the Principle is true with respect to the various compound machines described in Chapter XVII.

## Examples. XVIII.

1. On a Wheel and Axle a Power of 7 lbs. balances 1 cwt., and in one revolution of the Wheel the point of application of the Power moves through 32 inches: find through what height the Weight is raised.

2. The radius of the Wheel is 15 times that of the Axle, and when the Weight is raised through a certain height it is found that the point of application of the Power has moved over 7 feet more than the Weight: find the height through which the Weight was raised.

3. In the First System of Pullies find how much string passes through the hand in raising the Weight through one inch, there being four Pullies.

4. In the First System of Pullies it is found that 5 feet 4 inches of string must pass through the hand in order to raise the Weight 2 inches: find how many Pullies are employed.

5. In the First System of Pullies the distance of the highest Pully from the fixed end of the string which passes round it is 16 feet: find the greatest height through which the Weight can be raised, there being four Pullies.

6. In the Second System of Pullies if $P$ descends through 12 feet while $W$ rises through 2 feet, find the number of parts of the string at the lower block.

7. In the Second System of Pullies if there be five parts of the string at the lower block and $W$ rise through 6 inches, find how much $P$ descends.

8. In the Second System of Pullies if $P$ descend through 1 foot, $W$ will rise through $\frac{6}{n}$ inches, where $n$ is the number of Pullies in the lower block.

## XIX. *Friction.*

247. We have hitherto supposed that all bodies are *smooth;* practically this is not the case, and we must now examine the effect of the *roughness* of bodies.

248. The ordinary meaning of the words *smooth* and *rough* is well known; and a little explanation will indicate the sense in which these words are used in Mechanics.

Let there be a fixed plane horizontal surface formed of polished marble; place on this another piece of marble having a plane polished surface for its base. If we attempt to move this piece by a horizontal force we find that there is *some* resistance to be overcome: the resistance may be very small, but it always exists. We say then that the surfaces are not absolutely *smooth*, or we say that they are to some extent *rough.*

Thus surfaces are called *smooth* when no resistance is caused by them to the motion of one over the other; they are called *rough* when such a resistance is caused by them; and the resistance is called *friction.*

249. The following is another mode of defining the meaning of the words *smooth* and *rough* in Mechanics. When the mutual action between two surfaces in contact is perpendicular to them they are called *smooth;* when it is not perpendicular they are called *rough.*

When two surfaces in contact are both *plane* surfaces the definition is immediately applicable; when one or each of the surfaces is a *curved* surface some explanation is required. When one surface is curved and the other plane, the perpendicular to the plane surface at the point of contact is to be taken for the common perpendicular. When

each surface is curved, a plane must be supposed to touch each surface at the point of contact, and the perpendicular to this plane at the point of contact is to be taken for the common perpendicular.

250. The following laws relating to the *extreme amount* of friction which can be brought into action between plane surfaces have been established by experiment.

(1) *The friction varies as the normal pressure when the materials of the surfaces in contact remain the same.*

(2) *The friction is independent of the extent of the surfaces in contact so long as the normal pressure remains the same.*

These two laws are true not only when motion is just about to take place, but when there is sliding motion. But in sliding motion the friction is not always the same as in the state bordering on motion: when there is a difference the friction is greater in the state bordering on motion than in actual motion.

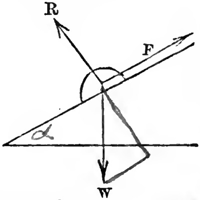
(3) *The friction is independent of the velocity when there is sliding motion.*

251. *Coefficient of friction.* Let $P$ denote the force perpendicular to the surfaces in contact by which two bodies are pressed together; and let $F$ denote the extreme friction, that is, let $F$ be equal to the force parallel to the surfaces in contact which is *just* sufficient to move one body along the other; then the ratio of $F$ to $P$ is called the *coefficient of friction.* It follows from the experimental laws stated in the preceding Article that the *coefficient of friction* is a constant quantity so long as we keep to the same pair of substances.

The following experimental results are given by Professor Rankine; they apply to actual motion:

The coefficient of friction for iron on stone varies between ·3 and ·7, for timber on timber varies between ·2 and ·5, for timber on metals varies between ·2 and ·6, for metals on metals varies between ·15 and ·25. Friction acts in the *opposite direction* to that in which motion actually takes place, or is about to take place.

252. *Angle of friction.* Let a body be placed on an Inclined Plane; if the Plane were perfectly smooth the body would not remain in equilibrium, but would slide or roll down the Plane: but practically owing to friction it is quite possible for the body to remain in equilibrium.




Let $W$ denote the weight of the body, which acts vertically downwards. Let $R$ denote the resistance of the Plane which acts at right angles to the Plane. Let $F$ denote the friction which acts along the Plane.

Suppose the body to be in equilibrium; then we have, as in Art. 215, by resolving the forces along the Plane and at right angles to it,

$$F - W \sin a = 0,$$
$$R - W \cos a = 0.$$

Hence by division, $\dfrac{F}{R} = \tan a$.

This result is true so long as the body is in equilibrium, whatever be the inclination of the Plane to the horizon. Now suppose the Plane to be nearly horizontal at first and let the inclination be gradually increased until the body is *just about* to slide down the Plane; the value of $\dfrac{F}{R}$ in this state is by our definition the *coefficient of friction:* so that the *coefficient of friction is equal to the tangent of the inclination of the Plane when the body is just about to slide.*

In this way we may experimentally determine the value of the coefficient of friction for any proposed pair of sur-

faces. The inclination of the Plane when the body is just about to slide is called the *angle of friction.*

253. The only case of friction besides that of *plane* surfaces which is practically important in Statics, is that of a hollow cylinder which can turn round a fixed cylindrical axis, and that of a solid cylinder which can turn within a fixed hollow cylinder or on a fixed plane. It is found by experiment that in these cases the extreme friction is very nearly proportional to the pressure; but the coefficient of friction is much less than it would be for plane surfaces of the same material kept in contact by the same pressure.

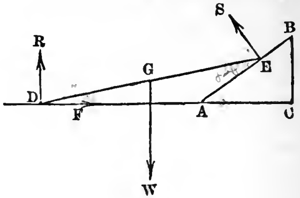
254. Statical Problems respecting friction involve two points, namely, the determination of the extreme or limiting position or positions *at* which equilibrium is possible, and the determination of the range of positions *within* which equilibrium is possible. Thus in Art. 252 the limiting position is that at which the tangent of the inclination of the Plane is equal to the coefficient of friction, and equilibrium will subsist as long as the inclination is less than the value thus determined. We may describe the process of solving statical problems involving friction thus: let $F$ denote a friction and $R$ the corresponding pressure; put $\mu R$ for $F$ in the conditions of equilibrium; then the limiting position of equilibrium is found by making $\mu$ equal to the *coefficient of friction* for the substances in consideration; and the range of positions within which equilibrium is possible is found by ascribing to $\mu$ values less than the coefficient of friction. If there are various frictions, and the pairs of surfaces in contact *not all of the same material*, we shall require different symbols to denote the ratio of the friction to the pressure in each case.

This general description will be illustrated by the remaining Articles of the present Chapter.

255. We will now solve the following problem:

One end of a uniform beam is on a smooth inclined plane, and the other end on a rough horizontal plane: determine the limiting positions of equilibrium.

Let $DE$ be the beam, $BA$ the inclined plane, $DAC$ the horizontal plane. Let $a$ denote the angle $BAC$, and $\theta$ the angle $EDA$.




The forces acting on the beam may be denoted as follows: a vertical force, $R$, and a horizontal force, $F$, arising from the action of the rough horizontal plane at $D$; a force, $S$, at right angles to the smooth inclined plane at $E$; and the weight of the beam, $W$, which acts vertically downwards through $G$, the centre of gravity of the beam.

We apply the conditions of equilibrium of Art. **93**. Resolve the forces vertically and horizontally: thus

$$R + S\cos a - W = 0 \quad\text{.....................(1)},$$

$$F - S\sin a = 0 \quad\text{.....................(2)}.$$

Take moments round $D$: thus

$$W.DG\cos\theta = S.DE\cos(a-\theta),$$

that is $$W\cos\theta = 2S\cos(a-\theta) \quad\text{..............(3)}.$$

Put $\mu R$ for $F$; then from (1) and (2)

$$\mu(W - S\cos a) = S\sin a,$$

therefore $$S = \frac{\mu W}{\sin a + \mu\cos a} \quad\text{......................(4)};$$

and then from (1) $$R = \frac{W\sin a}{\sin a + \mu\cos a} \quad\text{.....................(5)}.$$

Substitute the value of $S$ in (3): thus

$$\cos\theta = \frac{2\mu\cos(a-\theta)}{\sin a + \mu\cos a} \quad\text{......................(6)}.$$

The limiting position of equilibrium is assigned by the value of $\theta$ found from this equation, $\mu$ being put equal to the *coefficient of friction* which is supposed known. We proceed to investigate the range of positions of equilibrium. From (6) we obtain

$$\mu = \frac{\cos\theta \sin\alpha}{2\cos(\alpha-\theta) - \cos\alpha\cos\theta}$$

$$= \frac{\cos\theta\sin\alpha}{\cos\alpha\cos\theta + 2\sin\theta\sin\alpha} = \frac{\tan\alpha}{1+2\tan\theta\tan\alpha} \ldots(7).$$

Now it is obvious that $\theta$ must lie between 0 and $\alpha$. Hence we deduce the following results:

I. If the coefficient of friction is not less than $\tan\alpha$ every position is one of equilibrium.

II. If the coefficient of friction lies between $\tan\alpha$ and $\frac{\tan\alpha}{1+2\tan^2\alpha}$, then $\theta$ may have any value between $\alpha$ and the limiting position assigned by (6) or (7).

III. If the coefficient of friction is less than $\frac{\tan\alpha}{1+2\tan^2\alpha}$ there is no position of equilibrium.

In cases II. and III. suppose the beam in equilibrium when $\theta$ has an assigned value; and let $\mu$ be determined from (7): then (4) and (5) will determine the values of the forces $R$ and $S$, and (2) will determine the value of $F$.

The *resultant* action of the rough horizontal plane on the beam is the resultant of the forces $R$ and $F$; and is therefore equal to $\surd(R^2+F^2)$, that is to $\surd(1+\mu^2)\,R$ in the limiting position. This remark is applicable also in all similar cases.

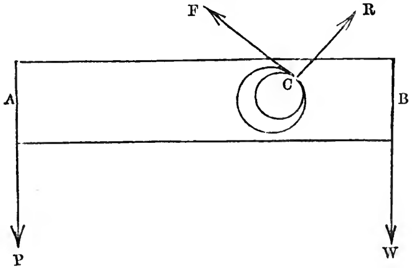
256. The beginner may perhaps object to the investigation of the preceding Article that it is impossible to obtain a perfectly smooth inclined plane, so that the pro-

blem cannot correspond with any real experience. We may reply, that although it is impossible to obtain a perfectly smooth inclined plane, yet there is no difficulty in imagining such a plane; and that the problem illustrates the principles of the subject very well even although the conditions which it supposes cannot be experimentally secured: and we may add, that it would be practically possible to make the inclined plane so very much smoother than the horizontal plane, that the friction arising from the former might be neglected in comparison with the friction arising from the latter.

We proceed to consider the effect of friction on some of the Mechanical Powers.

257. *The Lever with friction.*

Suppose a solid bar pierced with a cylindrical hole, through which passes a solid fixed cylindrical axis. Let the outer circle in the figure represent a section of the cylindrical hole made by a plane perpendicular to its axis; and let the inner circle represent the corresponding section of the solid axis. In the plane of this section we sup-




pose two forces, $P$ and $W$, to act on the bar at right angles to the bar. Also at $C$, the point of contact of the bar and the axis, there will be the action of the rough axis on the bar; denote this by a force $R$ along the common radius and a force $F$ along the common tangent. Suppose that these four forces keep the bar in equilibrium in a horizontal position.

Let $r$ denote the radius of the outer circle, $a$ and $b$ the lengths of the perpendiculars from the centre of this circle on the lines of action of $P$ and $W$ respectively. Let $\theta$ be the inclination of $CR$ to the vertical.

We apply the conditions of equilibrium of Art. 93. Resolve the forces parallel to the bar and at right angles to it: thus

$$R \sin\theta - F\cos\theta = 0 \quad\text{.................(1)},$$

$$R\cos\theta + F\sin\theta - P - W = 0 \quad\text{.................(2)}$$

Take moments round $C$: thus

$$P(a + r\sin\theta) = W(b - r\sin\theta) \quad\text{.................(3)}.$$

Put $\mu R$ for $F$; then in the limiting position of equilibrium $\mu$ is equal to the *coefficient of friction* which is supposed known: see Art. 253.

Thus from (1) $\sin\theta - \mu\cos\theta = 0$; therefore $\tan\theta = \mu$. This determines $\theta$; and then (3) determines the ratio of $P$ and $W$. From (1) and (2) we can find $R$ and $F$.

If the point of contact $C$ be supposed to fall on the other side of the vertical through the centre of the outer circle, we should have instead of (3),

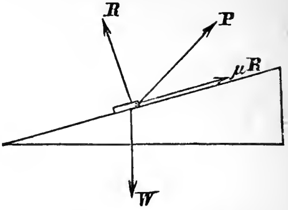
$$P(a - r\sin\theta) = W(b + r\sin\theta) \quad\text{.................(4)}.$$

From $\tan\theta = \mu$ we deduce $\sin\theta = \dfrac{\mu}{\sqrt{1+\mu^2}}$; and from (3) and (4) we infer that the bar will be in equilibrium provided the ratio of $\dfrac{P}{W}$ lies between

$$\frac{b - \dfrac{r\mu}{\sqrt{(1+\mu^2)}}}{a + \dfrac{r\mu}{\sqrt{(1+\mu^2)}}} \text{ and } \frac{b + \dfrac{r\mu}{\sqrt{(1+\mu^2)}}}{a - \dfrac{r\mu}{\sqrt{(1+\mu^2)}}},$$

where $\mu$ is the known coefficient of friction

258. *The Inclined Plane with friction.*




Let $a$ be the inclination of the Plane. Suppose a body of Weight $W$ placed on the Plane, and acted on by a Power $P$ in the direction which makes an angle $\beta$ with the Plane.

First suppose the body just on the point of moving *down* the Plane. Let $R$ denote the resistance which acts at right angles to the Plane, and $\mu R$ the friction which acts along the Plane *upwards*, $\mu$ being the coefficient of friction. Resolve the forces along the Plane, and at right angles to it, as in Art. 215: thus

$$P\cos\beta+\mu R-W\sin a=0 \qquad (1),$$

$$R+P\sin\beta-W\cos a=0 \qquad (2).$$

Substitute in (1) the value of $R$ from (2); thus

$$P=\frac{W\sin a-\mu W\cos a}{\cos\beta-\mu\sin\beta} \qquad (3).$$

Next suppose the body just on the point of moving *up* the Plane. The friction now acts *down* the Plane; and proceeding as before we obtain

$$P=\frac{W\sin a+\mu W\cos a}{\cos\beta+\mu\sin\beta} \qquad (4).$$

It will be seen that this result can be deduced from the former by changing the sign of $\mu$.

The equations (3) and (4) give the ratio of $P$ to $W$ in the limiting states of equilibrium; and we may infer that the body will be in equilibrium if $\frac{P}{W}$ lies between

$$\frac{\sin a-\mu\cos a}{\cos\beta-\mu\sin\beta} \text{ and } \frac{\sin a+\mu\cos a}{\cos\beta+\mu\sin\beta}.$$

A particular case of the general result may be noticed. Suppose $\beta=0$ so that the force acts along the Plane; then (3) and (4) give respectively

$$P=W(\sin\alpha-\mu\cos\alpha), \qquad P=W(\sin\alpha+\mu\cos\alpha).$$

Instead of an *Inclined* Plane suppose a body of weight $W$ placed on a rough *horizontal* Plane, and acted on by a force $P$ inclined at an angle $\beta$ to the horizon. Then we shall find that

$$P=\frac{\mu W}{\cos\beta+\mu\sin\beta}.$$

It will be seen that this result is the same as we should obtain by putting $\alpha=0$ in (4).

259. Let $\epsilon$ denote the *angle of friction;* see Art. 252. Then $\tan\epsilon=\mu$. The first value of $P$ of the preceding Article

$$=\frac{W\sin\alpha-\tan\epsilon\, W\cos\alpha}{\cos\beta-\tan\epsilon\sin\beta}=\frac{W\sin(\alpha-\epsilon)}{\cos(\beta+\epsilon)}.$$

The second value of $P$

$$=\frac{W\sin\alpha+\tan\epsilon\, W\cos\alpha}{\cos\beta+\tan\epsilon\sin\beta}=\frac{W\sin(\alpha+\epsilon)}{\cos(\beta-\epsilon)}.$$

Suppose we require to know the least Power which will suffice to prevent the body from moving down the Plane, the Power being allowed to act at the inclination to the Plane which we find most suitable.

Consider then that $\beta$ may vary: the first value of $P$ will be least when $\cos(\beta+\epsilon)=1$, that is when $\beta+\epsilon=0$, so that $\beta=-\epsilon$. Hence the Power must be supposed to act at an inclination to the Plane equal to the *angle of friction*, measured below the Plane. And the value of $P$ is $W\sin(\alpha-\epsilon)$.

Again, suppose we require to know the least Power which will suffice to move the body up the Plane. The second value of $P$ is least when $\cos(\beta-\epsilon)=1$, that is when $\beta=\epsilon$; and the value is then $W\sin(\alpha+\epsilon)$: this Power would keep the body on the point of motion up the Plane, and any greater Power would move the body.

As a particular case of the last result suppose $a=0$, so that instead of an Inclined Plane we have a horizontal Plane; thus we find that in order to move a given Weight along a rough horizontal Plane with the least Power we should make the Power act at an inclination to the Plane equal to the angle of friction; and then the body will be on the point of motion when the Power is equal to the Weight multiplied by the sine of the angle of friction.

260. *The Screw with friction.* See Art. 222.

Let $r$ be the radius of the cylinder, $b$ the length of the Power-arm, $a$ the angle of the Screw. Suppose that the Weight is about to prevail over the Power. Let $\mu$ be the coefficient of friction. The Screw is acted on by the following forces: the Weight $W$; the Power $P$; the resistances $R$, $S$, $T$,... at various points of the surfaces in contact, at right angles to the surfaces, and so all making an angle $a$ with the vertical; and frictions $\mu R$, $\mu S$, $\mu T$,... which will all make an angle $90^\circ - a$ with the vertical.

Then using the same conditions of equilibrium as in Art. 222, we have

$$W=(R+S+T+\ldots)\cos a+\mu(R+S+T+\ldots)\sin a \ldots\ldots\ldots(1),$$

$$Pb=(R+S+T+\ldots)\,r\sin a-\mu(R+S+T+\ldots)\,r\cos a \ldots.(2).$$

From (1) we obtain

$$R+S+T+\ldots=\frac{W}{\cos a+\mu\sin a},$$

then, substituting in equation (2)

$$Pb=\frac{r\sin a-\mu r\cos a}{\cos a+\mu\sin a}W;$$

therefore $$\frac{P}{W}=\frac{r(\sin a-\mu\cos a)}{b(\cos a+\mu\sin a)} \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If we suppose $P$ about to prevail over $W$ we obtain

$$\frac{P}{W}=\frac{r(\sin a+\mu\cos a)}{b(\cos a-\mu\sin a)} \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The equations (3) and (4) give the ratio of $P$ to $W$ in the limiting states of equilibrium; and we may infer that there will be equilibrium if $\frac{P}{W}$ lies between

$$\frac{r(\sin\alpha-\mu\cos\alpha)}{b(\cos\alpha+\mu\sin\alpha)} \text{ and } \frac{r(\sin\alpha+\mu\cos\alpha)}{b(\cos\alpha-\mu\sin\alpha)}.$$

If we put $\tan\epsilon$ for $\mu$ the expressions become

$$\frac{r}{b}\tan(\alpha-\epsilon) \text{ and } \frac{r}{b}\tan(\alpha+\epsilon).$$

261. If a body be placed on an Inclined Plane, and the friction be great enough to prevent sliding down the Plane, the body will stand or fall according as the vertical line drawn through the centre of gravity of the body falls within or without the base. This may be established in the manner of Art. 152.

## EXAMPLES. XIX.

1. Find the coefficient of friction if a Weight just rests on a rough Plane inclined at $45^0$ to the horizon.

2. A weight of 10 lbs. rests on a rough Plane inclined at an angle of $30^0$ to the horizon: find the pressure at right angles to the Plane and the force of friction exerted.

3. A force of 3 lbs. can, when acting along a rough Inclined Plane, just support a weight of 10 lbs., while a force of 6 lbs. would be requisite if the Plane were smooth: find the resultant pressure of the rough Plane on the Weight.

4. A body is just kept by friction from sliding down a rough Plane inclined at an angle of $30^0$ to the horizon: shew that no force acting along the Plane would pull the body upwards unless it exceeded the Weight of the body.

5. A body placed on a horizontal Plane requires a horizontal force equal to its own weight to overcome the friction: supposing the Plane gradually tilted, find at what angle the body will begin to slide.

6. In the preceding Example if additional support be given by means of a string fastened to the body and to a point in the Plane, so that the string may be parallel to the Plane, find at what inclination of the Plane the string would break, supposing the string would break on a smooth Inclined Plane at an inclination of $45^0$.

7. If the height of a rough Inclined Plane be to the length as 3 is to 5, and a Weight of 15 lbs. be supported by friction alone, find the force of friction in lbs.

8. If the height of a rough Inclined Plane be to the length as 3 is to 5, and a Weight of 10 lbs. can just be supported by friction alone, shew that it will be just on the point of being drawn up the Plane by a force of 12 lbs. along the Plane.

9. Find the force along the Plane required to draw a weight of 25 tons up a rough Inclined Plane, the coefficient of friction being $\frac{5}{12}$, and the inclination of the Plane being such that 7 tons acting along the Plane would support the Weight if the Plane were smooth.

10. Find the force in the preceding Example, supposing it to act at the most advantageous inclination to the Plane.

11. A ladder inclined at an angle of $60^0$ to the horizon rests between a *rough* pavement and the *smooth* wall of a house. Shew that if the ladder begin to slide when a man has ascended so that his centre of gravity is half way up, then the coefficient of friction between the foot of the ladder and the pavement is $\frac{1}{6}\sqrt{3}$.

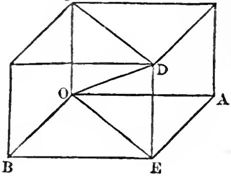
12. A heavy beam rests with one end on the ground, and the other end in contact with a vertical wall. Having given the coefficients of friction for the wall and the ground, and the distances of the centre of gravity of the beam from the ends, determine the limiting inclination of the beam to the horizon.

## XX. *Miscellaneous Propositions.*

262. In the present Chapter we shall give some miscellaneous propositions of Statics.

263. We have hitherto confined ourselves almost entirely to the equilibrium of forces which act in a plane: the following Articles will contain some propositions explicitly relating to forces which are not all in one plane.

264. *To find the resultant of three forces which act on a particle and are not all in one plane.*




Let $OA$, $OB$, $OC$ represent three forces in magnitude and direction which act on a particle at $O$. Let a parallelepiped be formed having these straight lines as edges; then the diagonal $OD$ which passes through $O$ will represent the resultant in magnitude and direction.

For $OE$, the diagonal passing through $O$ of the parallelogram $OAEB$, represents in magnitude and direction the resultant of the forces represented by $OA$ and $OB$. Also $OEDC$ is a parallelogram, and $OD$, the diagonal passing through $O$, represents in magnitude and direction the resultant of the forces represented by $OE$ and $OC$, that is, the resultant of the forces represented by $OA$, $OB$, and $OC$.

265. The preceding investigation is only a particular case of the general process given in Art. 52, but on account of its importance it deserves special notice. As we can thus compound three forces into one, so on the other hand we can resolve a single force into three others which act in assigned directions. Most frequently when we have thus to resolve a force the assigned directions are mutually at right angles; that is with the figure of Art. 264, the angles $AOB$, $BOC$, $COA$ are right angles. The angle $OCD$ is then a right angle, so that $OC = OD \cos COD$: thus *when the three components are mutually at right*

*angles a component is equal to the product of the resultant into the cosine of the angle between them.*

Also by Euclid, I. 47, we have

$$OD^2 = OC^2 + CD^2 = OC^2 + OE^2 = OC^2 + OB^2 + OA^2:$$

thus *when the three components are mutually at right angles the square of the resultant is equal to the sum of the squares of the three components.*

266. The process of Art. 52 for determining the resultant of any number of forces acting on a particle is applicable whether the forces are all in one plane or not; the process in Art. 56 assumes that the forces are all in one plane: we shall now extend the latter process to the case of forces which are not all in one plane.

267. *Forces act on a particle in any directions: required to find the magnitude and the direction of their resultant.*

Let $O$ denote the position of the particle; let $Op$, $Oq$, $Or$, $Os$,... denote the directions of the forces; let $P$, $Q$, $R$, $S$,... denote the magnitudes of the forces which act along these directions respectively. Draw through $O$ three straight lines mutually at right angles; denote them by $Ox$, $Oy$, $Oz$: and resolve each force into three components along these straight lines, by Art. 265. Thus $P$ may be replaced by the following three components: $P \cos pOx$ along $Ox$, $P \cos pOy$ along $Oy$, and $P \cos pOz$ along $Oz$. Similarly $Q$ may be replaced by $Q \cos qOx$ along $Ox$, $Q \cos qOy$ along $Oy$, and $Q \cos qOz$ along $Oz$. And so on.

Let $X$ denote the algebraical sum of the components along $Ox$; so that

$$X = P \cos pOx + Q \cos qOx + R \cos rOx + S \cos sOx + \dots;$$

Similarly let $Y$ and $Z$ denote the algebraical sums of the components along $Oy$ and $Oz$ respectively.

Thus the given system of forces is equivalent to the three forces $X$, $Y$, $Z$ which act along three straight lines mutually at right angles.

Let $V$ denote the resultant of the given system of forces, and $Ov$ its direction; then, by Art. 265,

$$V^2 = X^2 + Y^2 + Z^2,$$

$$\cos vOx = \frac{X}{V}, \quad \cos vOy = \frac{Y}{V}, \quad \cos vOz = \frac{Z}{V}.$$

Thus the magnitude and the direction of the resultant are determined.

268. Since $V \cos vOx = X$, *the resolved part of the resultant in any direction is equal to the sum of the resolved parts of the components in that direction:* see Arts. 44 and 52.

269. Suppose that any force $Q$ acts at a point $O$ in any direction $OF$; let $OD$ be any other direction: then the resolved part of $Q$ along $OD$ is $Q \cos FOD$.

Now let $OF$ make angles $\alpha$, $\beta$, $\gamma$ respectively with three straight lines $OA$, $OB$, $OC$ mutually at right angles; and let $OD$ make angles $\alpha'$, $\beta'$, $\gamma'$ respectively with $OA$, $OB$, $OC$. Then the force $Q$ may be resolved into the three forces $Q \cos \alpha$, $Q \cos \beta$, $Q \cos \gamma$ respectively along $OA$, $OB$, $OC$. Resolve each of these three components along $OD$; thus we obtain $Q \cos \alpha \cos \alpha'$, $Q \cos \beta \cos \beta'$, $Q \cos \gamma \cos \gamma'$ respectively. And as we may admit that the sum of these must be equal to the resolved part of $Q$ along $OD$, we have

$$Q \cos FOD = Q \cos \alpha \cos \alpha' + Q \cos \beta \cos \beta' + Q \cos \gamma \cos \gamma';$$

thus $\quad \cos FOD = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma'.$

Thus by the aid of statical considerations we arrive at the preceding formula which expresses the cosine of the angle between two straight lines in terms of the cosines of the angles which these straight lines make with three others mutually at right angles.

A particular case of the formula is obtained by supposing $OD$ to coincide with $OF$; then $\alpha' = \alpha$, $\beta' = \beta$, $\gamma = \gamma$: and we obtain

$$1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma.$$

270. In Art. 39 we have given the conditions under which three forces acting on a particle will maintain it in equilibrium; we will now present these conditions in a slightly different form, and then demonstrate a corresponding result for the case of four forces which are not all in one plane.

271. OA, OB, OC *are three straight lines of equal length in one plane, and they are not all on the same side of any straight line in the plane passing through* O; *forces* P, Q, R *respectively act along these straight lines such that*

$$\frac{P}{\text{area of } OBC} = \frac{Q}{\text{area of } OCA} = \frac{R}{\text{area of } OAB}:$$

*these forces will maintain a particle at* O *in equilibrium.*

For the area of a triangle is half the product of two sides into the sine of the included angle; hence each force is proportional to the sine of the angle between the directions of the other two: and the proposition follows immediately from Art. 39.

272. OA, OB, OC, OD *are four straight lines of equal length, no three of them being in the same plane, and they are not all on the same side of any plane passing through* O; *forces* P, Q, R, S *respectively act along these straight lines such that*

$$\frac{P}{\text{vol. } OBCD} = \frac{Q}{\text{vol. } OCDA} = \frac{R}{\text{vol. } ODAB} = \frac{S}{\text{vol. } OABC}:$$

*these forces will maintain a particle at* O *in equilibrium.*

Let $p$ denote the length of the perpendicular from $O$ on the plane $BCD$. Resolve each force into three along directions mutually at right angles, one direction being perpendicular to the plane $BCD$. The sum of the components of $Q$, $R$, and $S$ in the direction perpendicular to the plane $BCD$

$$= Q \times \frac{p}{OB} + R \times \frac{p}{OC} + S \times \frac{p}{OD} = (Q + R + S)\frac{p}{OB}.$$

Let $h$ denote the length of the perpendicular from $A$ on the plane $BCD$. The component of $P$ perpendicular to

the plane $BCD$ is $P\dfrac{h-p}{OA}$. Now the direction of the component of $P$ is *opposite* to the direction of the sum of the components of $Q$, $R$, and $S$ by reason of the condition that the straight lines $OA$, $OB$, $OC$, $OD$ are not all on the same side of any plane through $O$. Moreover by reason of the given ratios we have

$$\frac{P}{P+Q+R+S}=\frac{\text{vol. of } OBCD}{\text{sum of vols of } OBCD, OCDA, ODAB, OABC}$$
$$=\frac{\text{vol. of } OBCD}{\text{vol. of } ABCD}=\frac{p}{h};$$

therefore $\qquad Ph=(P+Q+R+S)p,$

and $\qquad P(h-p)=(Q+R+S)p.$

Thus the algebraical sum of the components perpendicular to the plane $BCD$ vanishes.

Similarly the algebraical sum of the components estimated perpendicular to $CDA$, $DAB$, and $ABC$ vanishes. Hence the resultant of the four forces vanishes; for if it did not the component estimated in all the four assigned directions could not vanish. See Art. 268.

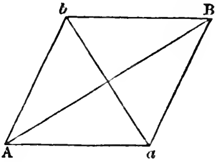
273. Conversely, if four forces acting on a particle maintain it in equilibrium, and no three of the forces are in the same plane, the forces must be in the proportion specified in the preceding Article.

For take $OA$, $OB$, $OC$, $OD$ all equal on the directions of the forces; then, resolving perpendicular to the plane $BCD$, we have as a necessary condition of equilibrium $\dfrac{P}{Q+R+S}=\dfrac{p}{h-p}$; thus $\dfrac{P}{P+Q+R+S}=\dfrac{p}{h}=\dfrac{\text{vol. of } OBCD}{\text{vol. of } ABCD}$: and similar expressions hold for the ratios of $Q$, of $R$, and of $S$, to $P+Q+R+S$.

274. We will now give some additions to our account of the theory of couples.

275. *Two unlike couples in parallel planes will balance if their moments are equal.*

Let there be two unlike couples in parallel planes of equal moments. By Art. 68 we may replace a couple by any like couple in the same plane which has an equal moment. Hence we may suppose the *forces* of one couple to be equal and parallel to the *forces* of the other couple: then as the moments are equal the *arms* of the couples will also be equal. Let $P$ and $p$ denote the forces of one couple; and $Q$ and $q$ the forces of the other; where $P$, $p$, $Q$, and $q$ are all numerically equal. Suppose $P$ and $Q$ to be like forces, and therefore $p$ and $q$ to be like. The resultant of $P$ and $Q$ will be $2P$, parallel to the direction of $P$ and $Q$, and midway between them. The resultant of $p$ and $q$ will be $2p$, parallel to the direction of $p$ and $q$, and midway between them.




Suppose any plane to cut the lines of action of $P$ and $p$ at $A$ and $a$ respectively, and of $Q$ and $q$ at $B$ and $b$ respectively. Then since the couples are in parallel planes $Aa$ is parallel to $Bb$; and since the arms are equal $Aa$ is equal to $Bb$. Thus $AaBb$ is a parallelogram. And since the couples are unlike, $A$ and $B$ are at the opposite ends of a diagonal. The resultant of $P$ and $Q$ acts at the middle point of $AB$, and the resultant of $p$ and $q$ at the middle point of $ab$; so that they act at the same point. And as the two resultants are equal but unlike they balance each other.

276. Hence *a couple is equivalent to another like couple of equal moment in any plane parallel to its own.*

277. We will briefly consider the case of two couples in two planes which are not parallel.

By Art. 68 we may transform each couple in its own plane until the two couples have a common arm situated in the straight line which is the intersection of the two planes. Let $Cc$ denote this common arm. Let $P$ and $p$ be the forces which form one couple, and $Q$ and $q$ the forces which

form the other; and suppose that $P$ and $Q$ act at $C$, and $p$ and $q$ at $c$. Then $P$ and $Q$ may be replaced by a single resultant $R$, and $p$ and $q$ by a single resultant $r$; also $R$ and $r$ will be equal and parallel but unlike. Thus the two couples are compounded into a single couple.

278. An example of Art. 97 which is of some interest may be noticed.

Let $A$, $B$, $C$......denote the positions of heavy particles in a plane; suppose at each of these points a force to act in the plane proportional to the product of the weight of the particle into its distance from a fixed point $O$, and at right angles to the distance; and suppose these forces all tend to turn the same way round $O$: it is required to replace the system of forces by a single force at $O$ and a couple.

First with respect to the single force. Suppose each force of the system to be $\mu$ times the product of the weight of the particle into the corresponding distance. If the direction of each force were *along* the corresponding distance instead of at right angles to it, the direction of the resultant would be *along* the straight line from $O$ to the centre of gravity of the particles, and the magnitude of the resultant would be $\mu$ times the product of the distance of the centre of gravity from $O$ into the sum of the weights of the particles: see Art. 154. Then in the actual case the magnitude of each force, supposed transferred to $O$, is the same as in the other case, but the direction is at right angles. Hence finally, in the actual case the direction of the single force is at right angles to the straight line drawn from $O$ to the centre of gravity; and the magnitude of the single force is $\mu$ times the product of the length of this straight line into the sum of the weights of the particles.

Next with respect to the couple. Let $P$ denote the weight of the particle at $A$; then the force at $A$ is $\mu P \times OA$; and the moment of this force round $O$ is $\mu P \times OA \times OA$, that is $\mu P \times OA^2$. Hence finally, the moment of the couple is $\mu$ times the sum of the product of the weight of each particle into the square of the corresponding distance from $O$.

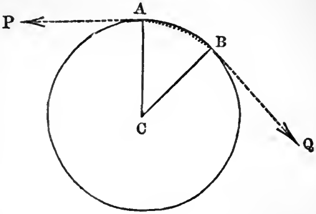
279. We have not discussed the case of a system of forces acting at any points and in any directions on a rigid body; but the investigations which have been given will enable the student to discuss such a case.

For suppose forces $P, Q, R, S,...$ to act at various points of a rigid body, and in various directions. Since a force may be supposed to act at any point in its line of action we may suppose these forces to be applied at points which are *all in one plane.* Then resolve each force into two components, one in the plane and one at right angles to the plane: thus we obtain two systems of forces, one in the plane and one at right angles to the plane. The former system if not in equilibrium will reduce to a single force or a couple: see Art. 84. The latter system if not in equilibrium will also reduce to a single force or a couple: see Art. 112. Hence in any particular case we shall be able to find the resultant of all the forces.

It is plain that the original system of forces will not be in equilibrium unless each of the two systems into which we have resolved it is separately in equilibrium; for one of the two cannot balance the other. Hence from considering the system at right angles to the plane we arrive at the following result: *if a system of forces is in equilibrium the algebraical sum of the forces resolved parallel to any straight line must vanish.* We say that this is *necessary* to equilibrium; we do not say that it is *sufficient.*

280. When we have spoken of a string passing round a peg or a pully we have hitherto assumed the peg or pully to be *smooth.* But in practice there may be a sensible amount of *roughness;* and every one must have observed that if a rope be coiled two or three times round a post, it is possible for a force at one end to balance a much larger force at the other end. This is owing to the friction between the rope and the post; and we shall now give some investigations relating to this subject.

281. *A string is stretched round a rough right circular cylinder in a plane perpendicular to the axis: to shew that as the portion of string in contact with the cylinder increases in Arithmetical Progression the mechanical advantage increases in Geometrical Progression.*




Suppose $AB$ the portion of the string in contact with the cylinder; and let $C$ be the centre of the circle of which $AB$ is an arc. Let $P$ and $Q$ be the tensions of the string at $A$ and $B$ respectively; and suppose $Q$ the greater. Suppose the string to be in the limiting condition of equilibrium, so that it is just about to move from $A$ towards $B$.

I. The relation between $P$ and $Q$ will be of the form $\frac{Q}{P}=K$, where $K$ is some quantity which does not depend on the forces.

For suppose that without changing the angle $ACB$, or the radius $AC$, or the coefficient of friction, we double $P$; then equilibrium will still hold if we also double $Q$. For the result is the same as if we had two strings in contact with the same cylinder, over equal arcs, and each acted on by a force $P$ at one end, and a force $Q$ at the other.

Similarly, if $P$ be changed to $3P$, and $Q$ to $3Q$, equilibrium will still hold; and so on. Thus if the angle, the radius, and the coefficient of friction remain the same, $Q$ varies as $P$.

II. Let the angle $ACB$ be denoted by $\theta$: then $K$ must be of the form $k^\theta$, where $k$ is some quantity which does not depend on the forces, nor on $\theta$.

For take any arbitrary angle $a$, and suppose that $\theta=na$. Imagine $AB$ to be divided into $n$ equal parts; and let $Q_1$, $Q_2$, $Q_3$,... be the tensions at the end of the first, second, third,... of these parts, beginning from $A$. Then by what has been already shewn, we have

$$\frac{Q_1}{P}=\frac{Q_2}{Q_1}=\frac{Q_3}{Q_2}=\dots\dots=H,$$

where $H$ is some quantity which does not depend on the forces, nor on $\theta$. Hence by multiplication

$$\frac{Q}{P}=H^n=H^{\frac{\theta}{a}}=k^{\theta},$$

where $k=H^{\frac{1}{a}}$; and $k$ does not depend on the forces, nor on $\theta$.

If the length of string in contact with the cylinder increases in Arithmetical Progression, then $\theta$ increases in Arithmetical Progression; and thus the ratio of $Q$ to $P$ increases in Geometrical Progression.

This result explains the very great mechanical advantage which is gained by coiling a rope two or three times round a post. Suppose, for example, that when a rope is coiled *once* round a post we have $\frac{Q}{P}=3$; then when the rope is coiled *twice* round the post $\frac{Q}{P}=3^2$; when the rope is coiled *three times* round $\frac{Q}{P}=3^3$; and so on.

282. We will now determine the value of $k$, as the process is instructive, although it requires more knowledge of mathematics than we have hitherto assumed.

The forces which act on the portion $AB$ of the string are the following: $P$ along the tangent at $A$, $Q$ along the tangent at $B$, and a resistance and a friction on every indefinitely small element of the string $AB$.

The resistance on every element is a force the direction of which passes through $C$: the corresponding friction is $\mu$ times this resistance and its direction is at right angles to that of the resistance. Suppose $R$ to denote the resultant of all the resistances, and $\phi$ to denote the angle its direction makes with $CA$; then $\mu R$ will denote the resultant of all the frictions, and its direction will make an angle $\phi$ with the tangent at $A$.

Suppose the string to be in equilibrium; if it were to become rigid, equilibrium would still subsist; the forces therefore must satisfy the conditions of Art. 93. Hence

resolving parallel to the tangent and to the radius at $A$ we have

$$P + \mu R \cos\phi = R \sin\phi + Q \cos\theta \text{.........(1)},$$
$$R \cos\phi + \mu R \sin\phi = Q \sin\theta \text{.........(2)}.$$

From (2) we have $R = \dfrac{Q \sin\theta}{\cos\phi + \mu \sin\phi}$;

substitute in (1); thus

$$P = Q\cos\theta + \frac{Q \sin\theta\,(\sin\phi - \mu\cos\phi)}{\cos\phi + \mu\sin\phi}.$$

Put $\tan a$ for $\mu$; thus

$$P = Q\cos\theta + \frac{Q\,(\sin\phi\cos a - \cos\phi\sin a)}{\cos\phi\cos a + \sin\phi\sin a}\sin\theta;$$

therefore $\qquad P = \{\cos\theta - \tan(a - \phi)\sin\theta\}\, Q.$

But $\dfrac{P}{Q} = \dfrac{1}{k^\theta} = k^{-\theta}$, so that $k^{-\theta} = \cos\theta - \sin\theta\tan(a - \phi)$;

therefore $\qquad \tan(a - \phi) = \dfrac{\cos\theta - k^{-\theta}}{\sin\theta}.$

This is an exact equation which is true for all values of $\theta$, and is therefore true when $\theta$ is indefinitely small: from this consideration we shall deduce the value of $k$. The value of $k$ will depend on the unit of angular measure which we adopt: we will take the unit to be that of circular measure. Now when $\theta$ is indefinitely small, so also is $\phi$, and the left-hand member of the last equation becomes $\tan a$. Also $\cos\theta = 1$, and $k^{-\theta} = 1 - \theta\log k$ very approximately, so that we have on the right-hand side of the equation $\dfrac{\theta\log k}{\sin\theta}$; and this by Trigonometry is equal to $\log k$, when $\theta$ is indefinitely small.

Thus $\log k = \tan a = \mu$, therefore $k = e^\mu$.

283. It will be seen that in the preceding Article we only had occasion to employ *two* out of the *three* equations of equilibrium of Art. 93. To form the third equation we will take moments round $C$: thus we find that $Q - P$ is equal to the sum of all the frictions exerted.

## EXAMPLES. XX.

1. Three forces of 11, 10, and 2 lbs. respectively act on a particle in directions mutually at right angles: determine the magnitude of the resultant.

2. Three forces $P$, $P$, and $P\sqrt{2}$ act on a particle in directions mutually at right angles: determine the magnitude of the resultant, and the angles between the direction of the resultant and that of each component.

3. Three forces each equal to $P$ act on a particle, and the angle between the directions of any two forces is $2\alpha$; if $R$ denote the resultant, and $\theta$ the angle between the direction of the resultant and that of each component, shew that

$$\sin\theta = \frac{2}{\sqrt{3}}\sin\alpha, \quad R^2 = P^2(9 - 12\sin^2\alpha).$$

4. A particle is placed at the corner of a cube, and is acted on by forces $P$, $Q$, $R$ along the diagonals of the faces of the cube which meet at the particle: determine the resultant force.

5. Two couples act in planes which are at right angles to each other; each force of one couple is 3 lbs., and the arm is one foot; each force of the other couple is 2 lbs., and the arm is two feet: determine the moment of the resultant couple.

6. $D$ is the vertex of a pyramid on a triangular base $ABC$; forces $P$, $Q$, $R$ act at the centres of gravity of the faces $DBC$, $DCA$, $DAB$, at right angles to these faces respectively, and such that

$$\frac{P}{\text{area of } DBC} = \frac{Q}{\text{area of } DCA} = \frac{R}{\text{area of } DAB}:$$

shew, by resolving $P$, $Q$, and $R$, parallel and perpendicular to the base $ABC$, that their resultant is perpendicular to $ABC$, and passes through the centre of gravity of $ABC$; and that the resultant bears the same ratio to the area of $ABC$ as $P$ bears to the area of $DBC$.

## XXI. *Problems.*

284. We will close the part of the work relating to Statics with some observations on the solution of Mechanical Problems.

285. Problems may be proposed which have been formed by combining some definition or principle in Mechanics with some theorem of Pure Mathematics, and which cannot be solved briefly and simply without the aid of that theorem. The results given in Art. 121 exemplify this remark; they are obtained by combining Euclid VI. 3 and VI. A with an elementary principle respecting the centre of parallel forces. It is obvious that in order to solve problems of this kind the student requires a knowledge of the most important theorems in Pure Mathematics, together with a readiness in selecting the appropriate theorem, which can only be acquired by practice.

286. On the other hand problems may be proposed which do not depend so much on a knowledge of Pure Mathematics as on a correct use of mechanical principles; and respecting this class of problems we may make a few general remarks.

If forces act in one plane on a rigid body three conditions must be satisfied for equilibrium. These conditions may be expressed in various forms, as we have shewn in Chapter VI. The most interesting problems, and at the same time the most difficult, are such as relate to a system of two or more bodies which are in contact or connected by hinges or strings. The beginner should pay great attention to the following statements:

When a system of bodies is in equilibrium each body of the system must be in equilibrium; and so the forces which act on each body must satisfy the conditions of equilibrium.

When two bodies are in contact some letter should be used to denote the mutual action between them; and the conditions of equilibrium will enable us to find the magni-

tude of the force. With respect to the direction of the mutual action see Art. 249.

We assume that if two bodies $A$ and $B$ are in contact the force which $A$ exerts on $B$ is equal and opposite to that which $B$ exerts on $A$: this principle is called the *equality of action and reaction*, and it may be admitted as an axiom.

If two bodies are connected by a string some letter should be used to denote the tension, and the value of the tension found from the conditions of equilibrium.

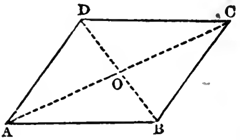
Beginners frequently make mistakes by *assuming* incorrect values for the action which takes place between two bodies, or for the tension of a string, instead of determining the values of these forces by the conditions of equilibrium.

When a body is in equilibrium under the action of forces in one plane *three* conditions of equilibrium must be satisfied, yet it may happen that we do not require to express all these conditions. For example, in the case of a Lever we may require only the *one* equation which is obtained by taking moments round the fulcrum; the other two equations would serve to determine the magnitude and the direction of the resistance of the fulcrum, and need not be formed if we do not wish to know these.

We shall illustrate these remarks by solving some problems.

287. Four beams without weight are connected by smooth joints so as to form a parallelogram; the opposite corners are connected by strings in tension: compare the tensions of the strings.

Let $ABCD$ represent the parallelogram. Let $P$ be the tension of the string $AC$, and $Q$ the tension of the string $BD$. Let $O$ be the intersection of $AC$ and $BD$.




The simplest mode of forming a joint is to pass a smooth

peg or pivot through the beams which are to be connected. Thus in the present case we have four beams and four pegs, and each of these must be in equilibrium. The strings may be supposed to join the pegs, and so not to be immediately connected with the beams.

Thus the beam $AB$ is acted on by only two forces, one from the peg at $A$, and the other from the peg at $B$. The two forces must therefore be equal and opposite, so that their line of action must coincide with $AB$. Denote each force by $R$.

Similarly $AD$ must be acted on by two equal and opposite forces, the line of action of which must coincide with $AD$. Denote each force by $S$.

The rod $AB$ exerts on the peg at $A$ a force $R$ equal and opposite to that which the peg exerts on the rod; similarly the rod $AD$ exerts on the peg at $A$ a force $S$ equal and opposite to that which the peg exerts on the rod. Thus the peg at $A$ is in equilibrium under the action of forces $P$, $R$, $S$ along $AC$, $BA$, and $DA$ respectively. Therefore, by Art. 38,

$$\frac{P}{S} = \frac{\sin DAB}{\sin BAC}.$$

In the same way by considering the equilibrium of the peg at $D$ we obtain

$$\frac{Q}{S} = \frac{\sin ADC}{\sin BDC}.$$

Therefore
$$\frac{P}{Q} = \frac{\sin BDC}{\sin BAC}$$

$$= \frac{\sin DBA}{\sin BAC} = \frac{AO}{BO} = \frac{AC}{BD}.$$

Thus the tensions of the strings are as the lengths of the diagonals along which they act.

288. It will be instructive to treat the preceding problem also in another manner.

The joint may be made by a peg or pivot which is rigidly attached to one beam and passes through the other. Suppose for example that the pegs are rigidly attached to the beams $AB$ and $CD$. We have then only four bodies to consider, namely the four beams. The strings may be supposed attached to the beams $AB$ and $CD$.

The beam $AD$ is acted on by forces at $A$ and $D$ arising from the other beams. The two forces must be equal and opposite, so that their line of action must coincide with $AD$. Denote each force by $S$.

Similarly the beam $BC$ is acted on by two forces which are equal and opposite, having $BC$ for their line of action. Denote each force by $T$.

The beam $AB$ is acted on by four forces; namely $P$ along $AC$, $Q$ along $BD$, $S$ having $AD$ for its line of action, and $T$ having $BC$ for its line of action.

We shall apply the conditions of equilibrium of Art. 88.

Take moments round $B$. Thus

$$S.AB \sin BAD = P.AB \sin BAO;$$

so that
$$S \sin BAD = P \sin BAO:$$

and $S$ must act on the beam $AB$ in the direction $DA$.

Take moments round $A$. Thus

$$T.AB \sin ABC = Q.AB \sin ABO;$$

so that
$$T \sin ABC = Q \sin ABO:$$

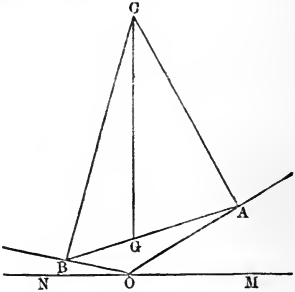
and $T$ must act on the beam $AB$ in the direction $CB$.

Take moments round $O$. Thus $T = S$.

Hence
$$\frac{\sin BAD}{\sin ABC} = \frac{P \sin BAO}{Q \sin ABO};$$

therefore
$$\frac{P}{Q} = \frac{\sin ABO}{\sin BAO} = \frac{AO}{BO}.$$

289. A heavy rod rests with its ends on two given smooth inclined planes: required the position of equilibrium.




Let $AB$ be the rod, $AOM$ and $BON$ the inclined planes; $MON$ being a horizontal line.

The forces acting on the rod are the resistance of the plane at $A$, at right angles to $OA$, the resistance of the plane at $B$, at right angles to $OB$, and the weight vertically downwards through the centre of gravity of the rod. Let $AC$ and $BC$ be the lines of action of the resistances of the planes, and $G$ the centre of gravity of the rod; then the directions of the three forces must meet at a point by Art. 41, so that $G$ must be vertically under $C$. Join $CG$.

Let $AG=a$, $BG=b$, the angle $AOM=\alpha$, and the angle $BON=\beta$; and let $\theta$ be the inclination of $AB$ to the horizon, that is, the angle between $AB$ and $MN$ produced.

Since $CA$ and $CG$ are perpendicular to $OA$ and $OM$ respectively, the angle $GCA=$the angle $AOM=\alpha$. Similarly the angle $GCB=\beta$.

The angle $OAB=\alpha-\theta$, and the angle $ABO=\beta+\theta$, by Euclid, I. 32.

Now
$$\frac{GC}{GA}=\frac{\sin GAC}{\sin GCA}=\frac{\cos(\alpha-\theta)}{\sin\alpha};$$
$$\frac{GC}{GB}=\frac{\sin GBC}{\sin GCB}=\frac{\cos(\beta+\theta)}{\sin\beta};$$
therefore, by division,
$$\frac{b}{a}=\frac{\sin\beta}{\sin\alpha}\,\frac{\cos(\alpha-\theta)}{\cos(\beta+\theta)};$$

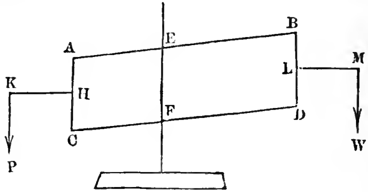
therefore $$\frac{b\sin\alpha}{a\sin\beta}=\frac{\cos(\alpha-\theta)}{\cos(\beta+\theta)}=\frac{\cos\alpha\cos\theta+\sin\alpha\sin\theta}{\cos\beta\cos\theta-\sin\beta\sin\theta}$$

$$=\frac{\cos\alpha+\sin\alpha\tan\theta}{\cos\beta-\sin\beta\tan\theta};$$

therefore $$\tan\theta=\frac{b\sin\alpha\cos\beta-a\sin\beta\cos\alpha}{(a+b)\sin\alpha\sin\beta}$$

$$=\frac{b\cot\beta-a\cot\alpha}{a+b}.$$

290. As another example we will explain the Balance called *Roberval's Balance.*




$AB$ and $CD$ are equal beams which can turn in a vertical plane round fixed points $E$ and $F$ in the same vertical line; $AE$ being equal to $CF$.

$AC$ and $BD$ are equal beams connected with the former beams by pivots at $A$, $B$, $C$, and $D$. $HK$ is a beam rigidly attached to $AC$, and $LM$ is a beam rigidly attached to $BD$; the angles $AHK$ and $BLM$ being right angles. A weight $P$ is hung at $K$ and a weight $W$ is hung at $M$: it is required to find the ratio of $P$ to $W$ when there is equilibrium, neglecting the weights of the beams.

Since $EF$ is vertical so are $AC$ and $BD$; and $KH$ and $LM$ are horizontal.

The piece formed of $KH$ and $AC$ is acted on by the weight $P$, by a force at $A$ arising from the beam $AB$, and by a force at $C$ arising from the beam $CD$. Resolve the force at $A$ into two components, one vertical and the other in the straight line $AB$. Resolve the force at $C$ into two components, one vertical and the other in the straight line $CD$.

The components in the straight lines $AB$ and $CD$ must be *equal and unlike;* for if they were not the sum of the horizontal components of the forces on the piece would not be zero.

Let $Y$ denote the vertical force on the piece at $A$, supposed upwards: then the vertical force on the piece at $C$ must be $P - Y$ upwards: for if it were not the sum of the vertical components of the forces on the piece would not be zero.

Thus $AB$ is acted on at $A$ by some force in the straight line $AB$, and by a vertical force $Y$ downwards. And $CD$ is acted on at $C$ by some force in the straight line $CD$, and by a vertical force $P - Y$ downwards. Similarly $AB$ is acted on at $B$ by some force in the straight line $AB$, and by a vertical force downwards which we may denote by $Z$. And $CD$ is acted on at $D$ by some force in the straight line $CD$, and by a vertical force $W - Z$ downwards.

Also $AB$ is acted on by some force at $E$, and $CD$ by some force at $F$.

Take moments round $E$ for $AB$: thus

$$Y \times EA \sin EAC = Z \times EB \sin EBD;$$

therefore $$Y \times EA = Z \times EB.$$

Take moments round $F$ for $CD$; thus

$$(P - Y)\, FC \sin FCA = (W - Z)\, FD \sin FDB;$$

therefore $$(P - Y)\, EA = (W - Z)\, EB.$$

Hence, by addition,

$$P \times EA = W \times EB.$$

Thus the ratio of $P$ to $W$ is independent of the lengths of $HK$ and $LM$; and if $EA = EB$ then $P = W$.

In practice $CA$ and $DB$ are produced vertically upwards, and have pans rigidly attached to them, in which $P$ and $Q$ are placed, instead of being hung at the ends of $KH$ and $LM$.

## *Miscellaneous Examples in Statics.*

1. The magnitudes of two bodies are as 3 is to 2, and their weights are as 2 is to 1: compare their densities.

2. Two forces act on a particle in directions at right angles to each other; they are in the ratio of 5 to 12, and their resultant is equal to 65 lbs.: find the forces.

3. Three forces represented by 24, 25, and 7 are in equilibrium when acting on a particle: shew that two of them are at right angles.

4. The resultant of two forces which act at right angles on a particle is 51 lbs.; one of the components is 24 lbs.: find the other component.

5. Two forces acting in opposite directions to one another on a particle have a resultant of 28 lbs.; and if they acted at right angles they would have a resultant of 52 lbs.: find the forces.

6. $ABC$ is a triangle and $D$ is the middle point of $BC$; three forces represented in magnitude and direction by $AB$, $AC$, $DA$ act on a particle at $A$: find the magnitude and the direction of the resultant.

7. Three forces 3, 4, 5 act on a particle in the centre of a square in directions towards three of the angles of the square: find the magnitude and the direction of the force which will keep the particle at rest.

8. Forces $\sqrt{3}+1$, $\sqrt{3}-1$, and $\sqrt{6}$ act on a particle: find the angles between their respective directions that there may be equilibrium.

9. Three forces, represented by those diagonals of three adjacent faces of a cube which meet, act at a point: shew that the resultant is equal to twice the diagonal of the cube.

10. A string passing round a smooth peg is pulled at each end by a force of 10 lbs., and the angle between the parts of the string on opposite sides of the peg is $120^0$: find the pressure on the peg, and the direction in which it acts.

11. Three smooth pegs are fastened in a vertical plane so as to form an isosceles triangle with the base horizontal and the vertex downwards, and the vertical angle is equal to $120^0$. A fine string with a weight $W$ attached to each end is passed under the lower peg and over the other two pegs. Find the pressure on each peg. Find also the vertical pressure on each peg.

12. Find a point within an equilateral triangle at which if a particle be placed it will be kept in equilibrium by three forces represented by the straight lines joining the point with the angular points of the triangle.

13. Forces represented in magnitude and direction by the diagonals of a parallelogram act at one of the angles: find the single force which will counteract them.

14. If $R$ be the resultant of two forces $P$ and $Q$ acting on a particle, and $S$ the resultant of $P$ and $R$, shew that the resultant of $S$ and $Q$ will be $2R$.

15. Three equal forces act at a point, in directions parallel to three consecutive sides of a regular hexagon: find the magnitude and the direction of the resultant.

16. Shew that if one of two forces acting on a particle be given in magnitude and position, and also the direction of their resultant, the locus of the extremity of the straight line representing the other force will be a straight line.

17. A weight is supported by two strings which are attached to it, and to two points in a horizontal straight line: if the strings are of unequal length, shew that the tension of the shorter string is greater than that of the longer.

18. Two weights of 3 lbs. and 4 lbs. respectively are connected by a string which is passed over two smooth pegs in the same horizontal straight line: find what weight must be attached to the string between the pegs in order that when the weights have assumed their position of equilibrium the string may be bent at right angles.

19. A weight is supported by two strings equally inclined to the vertical: shew that if instead of one of them we substitute a string pulling horizontally so as not to disturb the position of the other, the tension of the latter will be doubled.

20. Three forces act on a particle; the forces are 1 lb., 4 lbs., and 6 lbs. respectively, and the force of 4 lbs. is inclined at an angle of $60^0$ to each of the other forces: find the magnitude and the direction of their resultant.

21. Two couples acting along the sides of a parallelogram are in equilibrium: find the ratio of the forces.

22. A straight rod two feet in length rests in a horizontal position between two fixed pegs placed at a distance of three inches apart, one of the pegs being at the end of the rod; a weight of 5 lbs. is suspended at the other end of the rod: find the pressure on each of the pegs.

23. A bent Lever has equal arms making an angle of $120^0$: find the ratio of the weights at the ends of the arms when the Lever is in equilibrium with one arm horizontal.

24. A heavy bent Lever of which one arm is twice the length of the other, and of which the arms form a right angle, is suspended by its angle, the point of suspension being two feet above a horizontal table; the extremity of the longer arm is just close to the table when the Lever is in equilibrium by its own weight: find the height above the table of the extremity of the shorter arm.

25. The ends of a uniform rod are connected by strings with the ends of another uniform rod which is moveable about its middle point: shew that when the system is in equilibrium either the rods or the strings are parallel.

26. Two cylinders of the same diameter whose lengths are 1 foot and 7 feet respectively, and whose weights are in the ratio of 5 to 3 are joined together so as to form one cylinder: find the position of the fulcrum about which the whole will balance.

27. A uniform bar $1\frac{1}{2}$ feet in length and 4 lbs. in weight rests in a horizontal position upon a fulcrum 3 inches distant from one end: find what weight acting at this end will keep the rod at rest. Find also the pressure on the fulcrum.

28. Find the centre of gravity of four weights 1 lb., 2 lbs., 3 lbs., 4 lbs., placed at the angular points of a square.

29. If a quadrilateral be such that one of its diagonals divides it into two equal triangles, the centre of gravity of the quadrilateral is in that diagonal.

30. Having given the positions of three particles $A, B, C$, and the positions of the centre of gravity of $A$ and $B$, and of the centre of gravity of $A$ and $C$, find the position of the centre of gravity of $B$ and $C$.

31. A heavy right-angled triangle is suspended by its right angle, and the inclination of the hypotenuse to the horizon is $40^0$: find the acute angles of the triangle.

32. Two scale pans are suspended from the two ends of a straight Lever whose arms are as 3 is to 4, and an iron bar of 20 lbs. weight is laid on the scale pans, and will just reach from the one to the other: find what weight must be put into one scale to preserve equilibrium.

33. A uniform rod 3 feet long and weighing 6 ounces is held horizontally in the hand, being supported by means of a finger below the rod two inches from the end, and the thumb over the rod at the end: find the pressures exerted by the finger and thumb respectively.

34. On a uniform straight Lever weighing 5 lbs. and 5 feet in length, weights of 1, 2, 3, 4 lbs. are hung at the distances 1, 2, 3, 4 feet respectively from one end: find the position of the fulcrum on which the whole will rest.

35. A uniform stick 6 feet long lies on a table with one end projecting beyond the edge of the table to the extent of two feet; the greatest weight that can be suspended from the end of the projecting portion without destroying the equilibrium is 1 lb.: find the weight of the stick.

36. Two equal particles are placed on two opposite sides of a parallelogram: shew that their centre of gravity will remain in the same position, if they move along the sides through equal lengths in opposite directions.

37. A beam capable of moving about one end is kept in a position inclined to the horizon at an angle of $60^0$ by a string attached to the other end; the string is inclined to the horizon at an angle of $60^0$ in an opposite direction: compare the tension of the string with the weight of the beam.

38. Two strings have each one of their ends fixed to a peg, and the other to the ends of a uniform rod: when the rod is hanging in equilibrium, shew that the tensions of the strings are proportional to their lengths.

39. A sugar loaf whose height is equal to twice the diameter of its base stands on a table, rough enough to prevent sliding, one end of which is gently raised until the sugar loaf is on the verge of falling over: when this is the case find the inclination of the table to the horizon.

40. A beam ten stone in weight and ten feet long rests on two points distant four feet from each end: find the greatest weight which is unable to turn it over, on whatever point of the beam it be placed.

41. A heavy uniform beam of weight $W$ is supported in a horizontal position by two men, one at each end; and a weight $Q$ is placed at three-fifths of the beam from one end: find the weight supported by each man.

42. A heavy beam is made up of two uniform cylinders whose lengths are as 3 is to 2, and weights as 3 is to 5: determine the position of the centre of gravity.

43. Three weights of 2 lbs., 3 lbs., and 4 lbs. respectively, are suspended from the extremities and the middle point of a rod without weight: determine the point in the rod about which the three weights will balance. If the three weights be interchanged in all possible ways, find how many such points there will be.

44. Four weights of 3 ounces, 2 ounces, 4 ounces, and 7 ounces respectively are at equal intervals of 8 inches on a Lever without weight, two feet in length: find where the fulcrum must be in order that they may balance. If the Lever is uniform and weighs 8 ounces, find the position to which it would be necessary to shift the fulcrum.

45. A rod 8 feet long balances about its middle point with a weight of 5 lbs. at one end, and a weight of 4 lbs. at the other end. If the weight of 5 lbs. be removed it is found that the rod will then balance about a point 1 foot 8 inches nearer the other end. Find the weight of the rod.

46. A rod 11 inches long has a weight of 7 ounces at one end, and a weight of 8 ounces at the other end, and is found to be in equilibrium when balancing on a fulcrum 5 inches from the heavier weight. If the weights are interchanged the fulcrum must be shifted $\frac{11}{17}$ of an inch. Find the weight of the rod, and the position of its centre of gravity.

47. If any triangle be suspended from the middle point of its base, and likewise a plumb line from the same point, shew that the plumb line will pass through the vertex of the triangle. If now we place a weight equal to one third of the weight of the triangle at either angle of the base, shew that the triangle will assume a position such that all the angles will have their perpendicular distances from the plumb line equal.

48. A heavy triangle is hung up by the angle $A$, and the opposite side is inclined at an angle $\theta$ to the vertical: if $B$ be the smaller of the other two angles of the triangle shew that $2 \cot \theta = \cot B - \cot C$.

49. Find the centre of gravity of a uniform wire 16 inches long, bent so as to form three sides of a rectangle, the middle length being 6 inches. If the ends be brought together so as to form a triangle, shew that the centre of gravity will be $\frac{5}{16}$ of an inch nearer to the base.

50. A uniform plank 20 feet long and weighing 42 lbs. is placed over a rail; two boys, weighing respectively 75 lbs. and 99 lbs., stand on the plank, each one foot from the end: find the position of the rail for equilibrium. If the two boys change places, find where a third boy weighing 72 lbs. must stand so as to maintain equilibrium without shifting the plank on the rail.

51. Find the centre of gravity of a cube from one corner of which a cube whose edge is one-half the edge of the first has been removed.

52. A pyramid is cut from a cube by a plane which passes through the extremities of three edges that meet at a point: find the distance of the centre of gravity of the remainder of the cube from the centre of the cube.

53. Two forces of 6 and 8 lbs. respectively act at the ends of a rigid rod without weight 10 feet long; the forces are inclined respectively at angles of $30^0$ and $60^0$ to the rod: find the force which will keep the rod at rest, and the point at which its direction crosses the rod.

54. A Wheel and Axle have radii respectively 2 feet 4 inches, and 5 inches. Find the Power which will balance a Weight of 3 cwt.

55. In the Wheel and Axle, supposing the rope which supports the Power to pass over a fixed pully so as to be horizontal on leaving the Wheel, find what difference would be made in the pressures on the fixed supports of the machine.

56. Find the magnitude of the Weight in the second system of Pullies if it exceed the Power by 40 lbs., and there are 6 parts of the string at the lower Block.

57. In the single moveable Pully with parallel strings a weight of 100 lbs. is suspended from the block, and the end of the string in which the Power acts is fastened at the distance of 2 feet from the fulcrum to a straight horizontal Lever 5 feet long, the fulcrum being at one end: find the force which must be applied at the other end of the Lever to preserve equilibrium.

58. If the weights of the Pullies in the first system, commencing with the highest, be 1, 2, 5, 6 lbs. respectively, find what Power will sustain a Weight of 24 lbs.

59. A capstan has four spokes, each projecting 8 feet from the axis. The cylinder round which the rope is wound has a diameter of 7 inches, and the rope itself is half an inch thick. If four men exert a force of 60 lbs. each at the ends of the spokes, find the tension of the rope.

60. A weight of 56 lbs. rests on a rough Plane inclined at an angle of $45^0$ to the horizon: find the pressure at right angles to the plane.

61. A body whose weight is $\sqrt{2}$ lbs. is placed on a rough Plane inclined to the horizon at an angle of $45^0$. The coefficient of friction being $\frac{1}{\sqrt{3}}$, find in what direction a force of $(\sqrt{3}-1)$ lbs. must act on the body in order just to support it.

62. A uniform pole leans against a smooth wall at an angle of $45^0$, the lower end being on a rough horizontal plane: shew that the amount of friction required to prevent sliding is half the weight of the pole.

63. A rough Plane is inclined to the horizon at an angle of $60^0$: find the magnitude and the direction of the least force which will prevent a body weighing 100 lbs. from sliding down the Plane, the coefficient of friction being $\frac{1}{\sqrt{3}}$.

64. A triangular plate is suspended by three parallel strings attached to the three corners; one of the strings can bear a weight of 2 lbs. without breaking, and each of the other two can bear a weight of 1 lb. without breaking: find the point of the triangular plate on which a weight of 4 lbs. may be placed without breaking any of the strings.

65. $ABCD$ is a triangular pyramid, $O$ is a point within it; like parallel forces act at $A$, $B$, $C$, $D$ proportional respectively to the volumes of the triangular pyramids $OBCD$, $OCDA$, $ODAB$, $OABC$: shew that the centre of the parallel forces is at $O$.

66. Parallel forces act at the angular points of a triangular pyramid, each force being proportional to the area of the opposite face; shew that the centre of the parallel forces is either at the centre of the inscribed sphere, or at the centre of one of the escribed spheres.

67. Two equal spheres are strung on a thread, which is then suspended by its extremities so that its upper portions are parallel: find the pressure between the spheres.

68. Two uniform rods $AB$, $BC$ of similar material are connected by a smooth hinge at $B$, and have smooth rings at their other ends which slide upon a fixed horizontal wire: shew that in equilibrium the smaller rod is vertical.

69. A rod $AB$ is fixed at an inclination of $60^0$ to a vertical wall; and a heavy ring of weight $W$ slides along it. The ring is supported by a tight string attached to the wall. Shew that the tensions of this string, when the ring is respectively pulled up and pulled down the rod by a force $\frac{W}{4}$ acting along the rod, are as 1 is to 3.

70. Parallel forces $P$, $Q$, $R$, $S$ act at the angular points of a tetrahedron: determine the parallel forces which must act at the centres of gravity of the faces of the tetrahedron, so that the second system may have the same centre and the same resultant as the first.

71. Perpendiculars are drawn from the angles of a triangle on the opposite sides; and at the feet of these perpendiculars act parallel forces proportional to $\sin 2A$, $\sin 2B$, $\sin 2C$: shew that their centre coincides with the centre of parallel forces proportional to $\tan A$, $\tan B$, $\tan C$ at the angular points.

72. Two equal heavy rods of weight $W$ are joined by a hinge at one end, and connected at the other ends by a thread on which a weight $w$ is capable of sliding freely: the system is then placed with the hinge resting on a horizontal plane, so that the two rods are in a vertical plane: shew that in the position of equilibrium the hanging weight cuts the vertical between the hinge and the horizontal straight line through the extremities of the rods in the ratio of $W$ to $w$.

73. Three equal rods $AB$, $BC$, $CD$ without weight, connected by hinges at $B$ and $C$, are moveable about hinges at $A$ and $D$, the distance $AD$ being twice the length of each rod. A force $P$ acts at the middle point of each of the rods $AB$ and $CD$, and at right angles to them: shew that the pressure on each of the hinges $A$ and $D$ will be $\frac{P}{\sqrt{3}}$, and that its direction will make an angle of $60^0$ with $AB$.

74. Two weights support each other on a rough double Inclined Plane by means of a fine string passing over the vertex, and no friction is called into operation: shew that

the Plane may be tilted about either extremity of the base through an angle $2\epsilon$ without disturbing the equilibrium, $\epsilon$ being the angle of friction, and both angles of the Plane being less than $90^0 - \epsilon$.

75. A Lever without weight is $c$ feet in length, and from its ends a weight is supported by two strings in length $a$ and $b$ feet respectively: shew that the fulcrum must divide the Lever into two parts, the ratio of which is that of $a^2+c^2-b^2$ to $b^2+c^2-a^2$, if there be equilibrium when the Lever is horizontal.

76. A uniform rod rests with one extremity against a rough vertical wall, the other extremity being supported by a string three times the length of the rod, attached to a point in the wall; the coefficient of friction is $\frac{7}{3}$: shew that the tangent of the angle which the string makes with the wall in the limiting position of equilibrium is $\frac{5}{27}$ or $\frac{1}{3}$.

77. If when two particles are placed on a rough double Inclined Plane, and connected by a string passing over a smooth peg at the vertex, they are on the point of motion, and when their positions are interchanged, no friction is called into play, shew that the angle of friction is equal to the difference of the inclinations of the two Planes.

78. A plane equilateral pentagon is formed of five equal uniform rods $AB, BC, CD, DE, EA$ loosely jointed together. The angular points $B$, $D$ of the pentagon are capable of sliding on a smooth horizontal rod, and the plane of the pentagon is vertical, the point $C$ being uppermost. Shew that if $\theta, \phi$ be the respective inclinations of the rods $AB$, $BC$ to the horizon in the position of equilibrium, $2 \tan \phi = \tan \theta$.

79. A uniform wire is formed into a triangle $ABC$, the lengths of the sides of which are $a$, $b$, $c$ respectively: shew that if $x$, $y$, $z$ be the respective distances of the centre of gravity of the wire from the middle points of its sides,

$$4(ax^2+by^2+cz^2)=abc.$$

80. If a particle be in equilibrium under the action of four equal forces, tending to the angular points of a tetrahedron, prove that the three straight lines passing through

the point, and through each pair of opposite edges of the tetrahedron are at right angles to each other.

81. Two weights are connected by a fine inextensible string which passes over a Pully; and one rests on a rough Inclined Plane, while the other hangs freely; if the string make angles $\theta_1$, $\theta_2$ with the Plane in the highest and lowest positions of equilibrium of the free weight, and $\theta$ when no friction is called into play, shew that

$$\cos\theta_2 + \cos\theta_1 - 2\cos\theta = \mu(\sin\theta_2 - \sin\theta_1),$$

where $\mu$ is the coefficient of friction.

82. A cylinder open at the top, stands on a horizontal plane, and a uniform rod rests partly within the cylinder, and in contact with it at its upper and lower edges in a vertical plane containing the axis of the cylinder: supposing the weight of the cylinder to be $n$ times that of the rod, find the length of the rod when the cylinder is on the point of tumbling.

83. Two equal rough balls lie in contact on a rough horizontal table; another ball is placed upon them so that the centres of the three are in a vertical plane: find the least coefficient of friction between the upper and lower balls and between the lower balls and the table, in order that the system may be in equilibrium.

84. Two uniform beams of equal weight but of unequal length, are placed with their lower ends in contact on a smooth horizontal plane, and their upper ends against smooth vertical planes: shew that in the position of equilibrium the two beams are equally inclined to the horizon.

85. A bowl is formed from a hollow sphere of radius $a$; it is so fixed that the radius of the sphere drawn to each point in the rim makes an angle $\alpha$ with the vertical, and the radius drawn to a point $A$ of the bowl makes an angle $\beta$ with the vertical: if a smooth uniform rod remains at rest when placed with one extremity at $A$, and with a point in its length on the rim of the bowl, shew that the length of the rod is $4a\sin\beta\sec\frac{1}{2}(\alpha-\beta)$.

# DYNAMICS.

## I. *Velocity.*

1. DYNAMICS treats of force producing or changing the motion of bodies.

Before we consider the influence of force on the motion of bodies we shall make some remarks on motion itself: we confine ourselves to the case of motion in a straight line.

2. The *velocity* of a point in motion at any instant is the degree of quickness of the motion of the point at that instant.

3. If a point in motion describe equal lengths of path in equal times the velocity is called *uniform* or *constant.* Velocity which is not uniform is called *variable.*

4. Uniform velocity is measured by the length of path described in the unit of time. We may take any unit of time we please; and a second is usually chosen. We may also take any unit of length we please: and a foot is usually chosen. Thus by the velocity 16 we mean the velocity of a point which moves uniformly in such a manner that the length of path described in one second is sixteen feet. The word *space* is used as an abbreviation of the term *length of path:* thus in the example just given it would be said that the *space* described in one second is sixteen feet.

5. *If a point moving with the uniform velocity* v *describe the space* s *in the time* t, *then* s = vt.

For in one unit of time $v$ units of space are described, and therefore in $t$ units of time $vt$ units of space are described; therefore $s = vt$.

6. Variable velocity is measured at any instant by the space which would be described in a unit of time, if the velocity were to continue during that unit of time the same as it is at the instant considered.

Hence, as in Art. 5, if $v$ denote the measure of a variable velocity at any instant, a point moving for the time $t$ with this velocity would describe the space $vt$.

7. The mode of measuring variable velocity is one with which we are familiar in practice. Thus a railway train may be moving with variable velocity, and yet we may say that at a certain instant it is moving at the rate of 30 miles an hour; we mean that if the train were to continue to move for one hour with just the same speed as at the instant considered it would pass over 30 miles.

8. The illustration just employed suggests that a velocity may be given expressed in any units of time and space; it is easy to express the velocity in terms of the standard units.

For example, suppose that a body is moving at the rate of 30 miles an hour. The body here is moving at the rate of $30 \times 5280$ feet in an hour, that is, in $60 \times 60$ seconds: hence it is moving at the rate of $\frac{30 \times 5280}{60 \times 60}$ feet in one second, that is, at the rate of 44 feet in one second. Hence 44 denotes the velocity when expressed in the standard units.

In like manner we may pass from the standard units to any other units.

For example, if $v$ denote a velocity when a second is taken as the unit of time, the *same* velocity will be denoted by $60v$ when a minute is taken as the unit of time. For to

say that a body is moving at the rate of $v$ feet per second is equivalent to saying that it is moving at the rate of $60v$ feet per minute.

In like manner if we wish to take a yard for the unit of length instead of a foot, as well as a minute for the unit of time instead of a second, the velocity denoted by $v$ with the standard units will now be denoted by $\frac{60v}{3}$.

Generally, let $v$ denote a velocity when a second is the unit of time, and a foot is the unit of length; then if we take $m$ seconds as the unit of time, and $n$ feet as the unit of length, the *same* velocity will be denoted by $\frac{mv}{n}$.

## EXAMPLES. I.

1. Compare the velocities of two points which move uniformly, one through 5 feet in half a second, and the other through 100 yards in a minute.

2. Compare the velocities of two points which move uniformly, one through 720 feet in one minute, and the other through $3\frac{1}{2}$ yards in three quarters of a second.

3. Two points move uniformly with such velocities that when they move in the *same* direction the distance between them increases at the rate of 5 feet per second, and when they move in *opposite* directions the distance between them increases at the rate of 25 feet per second: find the velocity of each.

4. A railway train travels over 100 miles in 2 hours: find the average velocity referred to feet and seconds.

5. One point moves uniformly round the circumference of a circle, while another point moves uniformly along the diameter: compare their velocities.

6. One point describes the circumference of a circle of $a$ feet radius in $b$ minutes; and another point describes the circumference of a circle of $b$ feet radius in $a$ minutes: compare their velocities.

## II. *The First and Second Laws of Motion.*

9. The science of Dynamics rests on certain principles which are called *Laws of Motion.* Newton presented them in the form of three laws; and we shall follow him.

It is not to be expected that a beginner will obtain a clear and correct idea of these laws on reading them for the first time; but as he proceeds with the subject and observes the applications of the laws he will gradually discover their full import. In like manner a beginner of geometry rarely comprehends at first all that is meant by the definitions, postulates, and axioms; but the imperfect notions with which he starts are corrected and extended as he studies the propositions.

In the present Chapter we shall chiefly discuss the First Law of Motion.

10. *First Law of Motion. Every body continues in a state of rest or of uniform motion in a straight line, except in so far as it may be compelled to change that state by force acting on it.*

It is necessary to limit the meaning of the word *motion* in the First Law. By the motion of a body is here meant that kind of motion in which every point of the body describes a straight line; in other words, there is to be no *rotation.* The rotation of bodies is discussed in works which treat of the highest branches of dynamics, and many important results are demonstrated: for example, it is shewn that if a free sphere of uniform density be rotating about a diameter at any instant, it will continue to rotate about that diameter if no force act on it.

In order to exclude all notion of rotation, some writers use the word *particle* instead of *body* in enunciating the First Law of Motion.

We must now proceed to consider the grounds on which we rest our belief in the truth of the First Law of Motion.

11. Little direct experimental evidence can be brought forward in favour of the truth of the Law. It is in fact impossible to preserve a body which is in motion from the action of external forces; and so it is impossible to obtain that perseverance in uniform motion of which the Law speaks. If we start a stone to slide along the ground we find that the stone is soon reduced to rest; but we have no difficulty in perceiving that the destruction of motion is due mainly to the friction of the ground. Accordingly we find that if the same stone is started with the same velocity to slide on a smooth sheet of ice, it will proceed much farther before it is reduced to rest. And we may easily imagine that if all such external forces as friction of the ground and resistance of the air were removed the motion would continue permanently unchanged.

In this illustration we suppose the stone to *slide* along the ground; we do not suppose the stone to *roll*, for the reason which is assigned in Art. 10.

12. But although the direct experimental evidence of the truth of this and of the other Laws of Motion is weak, the indirect evidence is very strong. For on these laws as a foundation the whole science of dynamics rests; the theory of astronomy forms a part of dynamics, and it is a matter of every day experience that the calculations and predictions of astronomy are most closely verified by observation. It seems in the highest degree improbable that numerous and intricate results, deduced from untrue laws, should be uniformly true; and accordingly we say that the agreement of theory and observation in astronomy justifies us in accepting the Laws of Motion.

13. From the First Law of Motion then we see that a body has no power to put itself in motion, or to change its motion; but a commencement or change of motion must be ascribed to the action of some external force.

14. It will be readily conjectured from common experience, that the effect of a given force in communicating or changing motion may depend partly on the size and the

kind of the body to which the force is applied; and this point will be discussed hereafter, so that we shall be able to compare the effect of a force on one body with the effect of the same force on another body. But at present we confine ourselves to the case in which a given force acts on a given body, so that we have only to consider the influence of the force on the *velocity* of the body.

15. *Second Law of Motion. Change of motion is proportional to the acting force, and takes place in the direction of the straight line in which the force acts.*

This law will require to be developed in order to place before the student all which its concise statement includes; but this development we shall reserve, as at present we only require a part of the law. We suppose a body in motion in a straight line, and acted on by a force in the direction of that straight line. Then we require so much of the Second Law of Motion as to enable us to assume that a given force communicates the same velocity in a given time, whatever be the velocity which the body already has. This is in fact included in the first clause of the Law: *change of motion is proportional to the acting force.* The whole meaning of this clause will be exhibited hereafter: see Art. 84.

It was scarcely necessary to introduce here even this brief notice of the Second Law of Motion; but without it the definition of *uniform force* which is given in the next Chapter might appear arbitrary and unnatural.

Although the student must not consider that he has mastered the subject until he understands the Laws of Motion, yet it is by no means necessary to weary himself by trying to understand these Laws fully before he passes on to any results deduced from them. He will learn more by examining the way in which these Laws are applied than by confining himself to the Laws themselves.

## Examples. II.

1. Two bodies start together from the same point and move uniformly along the same straight line in the same direction; one body moves at the rate of 15 miles per hour, and the other body at the rate of 18 feet per second: determine the distance between them at the end of a minute.

2. If the bodies move with the velocities of the preceding Example but in *opposite* directions, find when they will be 200 feet apart.

3. A body starts from a point and moves uniformly along a straight line at the rate of 30 miles per hour. At the end of half a minute another body starts from the same point after the former body, and moves uniformly at the rate of 55 feet per second. Find when and where the second body overtakes the first.

4. Two bodies start together from the same point and move uniformly in directions at right angles to each other; one body moves at the rate of 4 feet per second, and the other body at the rate of 3 feet per second: determine the distance between them at the end of $n$ seconds.

5. Supposing the earth to be a sphere 25000 miles in circumference, and turning round once in a day, determine the velocity of a point at the equator.

6. A mill sail is 7 yards long, and is observed to go round uniformly ten times in a minute: find the velocity of the extremity of the sail.

7. Two bodies start from the same point and move uniformly with the same velocity along straight lines inclined at an angle of $60^0$: find their distance apart at the end of a given time.

8. Two bodies start from the same point and move uniformly along straight lines inclined at an angle $a$: if the velocity of one body be $u$ and the velocity of the other body $v$, find their distance apart at the end of $n$ seconds.

## III. *Motion in a straight line under the influence of a uniform force.*

16. We confine ourselves to the case of motion in a straight line, and the direction of the force is supposed to be in the same straight line as that of the motion; and we consider only the effect of the force on the velocity without regard to the size and the kind of the body moved.

17. If a force acting on a body adds equal velocities in equal times, the force is called *uniform* or *constant.* Force which is not uniform is called *variable.*

18. Uniform force acting on a given body is measured by the velocity which is added in each successive unit of time. Variable force acting on a given body is measured at any instant by the velocity which would be added in a unit of time, if the force were to continue during that unit the same as it is at the instant considered.

19. We are now about to give some propositions respecting uniform force acting on a given body. The term *acceleration* is used as an abbreviation for the *velocity added in a unit of time;* so that when we speak of an acceleration $f$, we mean that by the action of a given force on a given body the velocity $f$ is added in a unit of time.

20. *A uniform force acts on a body in a fixed direction during the time* $t$: *if* $f$ *be the acceleration, and* $v$ *the velocity generated, then* $v = ft$.

For by the definition of uniform force, in each unit of time the velocity $f$ is communicated to the body; and therefore in $t$ units of time the velocity $ft$ is communicated.

21. *A body starting from rest is acted on by a uniform force in a fixed direction: if* $f$ *be the acceleration, and* $s$ *the space described in the time* $t$, *then* $s = \frac{1}{2} ft^2$.

Let the whole time $t$ be divided into $n$ equal intervals; denote each interval by $\tau$, so that $n\tau = t$.

Then the velocity of the body at the end of the times

$$\tau,\ 2\tau,\ 3\tau,\ \ldots\ldots(n-1)\tau,\ n\tau$$

from starting, is, by Art. 20, respectively,

$$f\tau,\ 2f\tau,\ 3f\tau,\ \ldots\ldots(n-1)f\tau,\ nf\tau.$$

Let $s_1$ denote the space which the body would describe if it moved during each interval $\tau$ with the velocity which it has at the *beginning* of the interval; and let $s_2$ denote the space which the body would describe if it moved during each interval $\tau$ with the velocity which it has at the *end* of the interval. Then

$$s_1 = 0 \cdot \tau + f\tau \cdot \tau + 2f\tau \cdot \tau + 3f\tau \cdot \tau + \ldots + (n-1)f\tau \cdot \tau,$$

$$s_2 = f\tau \cdot \tau + 2f\tau \cdot \tau + 3f\tau \cdot \tau + \ldots\ldots\ldots\ldots + nf\tau \cdot \tau;$$

that is,

$$s_1 = f\tau^2\{1+2+3+\ldots+(n-1)\},$$

$$s_2 = f\tau^2\{1+2+3+\ldots+(n-1)+n\}.$$

Hence, by the theory of Arithmetical Progression in Algebra, we have

$$s_1 = f\tau^2\frac{(n-1)\,n}{2} = \frac{ft^2}{n^2}\cdot\frac{(n-1)\,n}{2} = \frac{ft^2}{2}\left(1-\frac{1}{n}\right),$$

$$s_2 = f\tau^2\frac{n\,(n+1)}{2} = \frac{ft^2}{n^2}\cdot\frac{n\,(n+1)}{2} = \frac{ft^2}{2}\left(1+\frac{1}{n}\right).$$

Now $s$, the space actually described, must lie between $s_1$ and $s_2$; but by making $n$ large enough we can make $\frac{1}{n}$ as small as we please; so that we can make $s_1$ and $s_2$ differ from $\frac{1}{2}ft^2$ by less than any assigned quantity. Hence $s = \frac{1}{2}ft^2$.

22. Thus if a body start from rest and be acted on for the time $t$ in a fixed direction by a uniform force of which the acceleration is $f$, we have the following values, of $v$ the velocity acquired, and $s$ the space described,

$$v = ft \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$s = \frac{1}{2} ft^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) and (2) by eliminating $t$ we have

$$v^2 = 2fs \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

this gives the velocity in terms of the acceleration and the space described.

We may of course modify the forms of these expressions by common Algebra; for example, (2) may be written thus:

$$t = \sqrt{\frac{2s}{f}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (1) and (2) we may deduce

$$s = \frac{1}{2} vt \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

this shews that the space actually described is half that which would be described by a body moving for the time $t$ with a uniform velocity equal to $v$.

These formulæ are very important, and will often be applied.

23. *Falling bodies.* When bodies are allowed to fall freely to the surface of the earth from heights above it, we find that different bodies fall through *equal* spaces from rest in a given time, and that the space fallen through in any time from rest varies as the *square* of the time. These laws at least hold approximately, and the resistance of the air appears to be the reason of such deviations from exact conformity with these laws as may be observed. For example, a sovereign and a feather do not fall to the ground in the same time if the experiment be tried in the

open air; but they do if the experiment be tried in the exhausted receiver of an air-pump.

From these observed facts, compared with the results given in the preceding Article, we infer that the Earth exerts a force in the vertical direction on all bodies, that this force is a uniform force, and that it produces the same acceleration in all bodies. This force is called *gravity*.

24. The letter $g$ is invariably used to denote the acceleration produced by gravity.

It is found that the value of $g$ increases slightly as we pass from the equator towards the poles. At London $g=32.19$ feet nearly, the unit of time being one second. That is, when a body falls freely in the latitude of London a velocity of 32.2 feet nearly is communicated to it every second.

Moreover, the value of $g$ is not the same at different heights above the same point of the Earth's surface; the force which the Earth exerts on a given body varies very nearly inversely as the square of its distance from the centre of the Earth. But as any heights to which we can ascend are very small compared with the radius of the Earth, the change thus produced in the force of gravity will also be very small.

The direction of the force of gravity is perpendicular to the horizontal plane at every place, and so really varies from point to point on the Earth's surface. But this variation will be scarcely sensible so long as we do not move more than a few miles from an assigned spot.

Thus on the whole we may practically, in the vicinity of any assigned spot, regard the direction of the force of gravity as parallel to the same straight line, and the value of $g$ as constant.

25. The laws respecting the variation of the force of gravity which we have stated in the preceding Article are suggested by observation and experiment; their exact truth is established by shewing that results deduced by calculation from these assumed laws are verified in numerous cases; see Art. 12.

26. Thus, by Art. 22, when bodies fall from rest the following formulæ apply:

$$v=gt\ldots(1);\quad s=\frac{1}{2}gt^2\ldots(2);\quad v^2=2gs\ldots(3).$$

For example, take the second formula, and put for $t$ in succession 1, 2, 3, 4,...; thus we obtain the following results: in 1, 2, 3, 4,... seconds respectively from rest the spaces described are $\frac{1}{2}g, \frac{4}{2}g, \frac{9}{2}g, \frac{16}{2}g\ldots$. Hence by subtracting each of these numbers from that which follows it, we find that the spaces described in the *second*, *third*, *fourth*,... seconds respectively are $\frac{3}{2}g, \frac{5}{2}g, \frac{7}{2}g,\ldots$.

And generally in $n-1$ seconds the space described is $\frac{1}{2}(n-1)^2g$; in $n$ seconds the space described is $\frac{1}{2}n^2g$; hence *during the* $n^{\text{th}}$ *second* the space described is

$$\frac{1}{2}n^2g-\frac{1}{2}(n-1)^2g, \text{ that is, } \frac{1}{2}(2n-1)\,g.$$

The velocity acquired by a body falling from rest through the space $s$ is sometimes called *the velocity due to the space* s *under the action of gravity;* and the space through which a body must fall from rest to acquire a velocity $v$ is sometimes called *the space due to the velocity* $v$ under the action of gravity.

27. *Motion down an Inclined Plane.* We may now consider the motion of a body sliding down a smooth Inclined Plane.

Suppose there is a smooth Plane inclined at an angle $a$ to the horizon. The force which the earth exerts on a body acts in the vertical direction; we may resolve this force into two components, one along the Plane, and the other perpendicular to the Plane. The component along the Plane is obtained by multiplying the whole force by $\sin a$, and so we may naturally *assume* that the acceleration due to this component is obtained by multiplying the whole acceleration by $\sin a$: thus, this acceleration is $g \sin a$. The force perpendicular to the Plane has no influence on the motion down the Plane; it is counteracted by the resistance of the Plane.

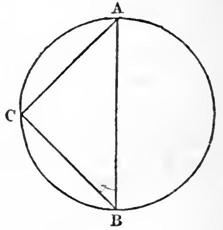
Hence we conclude that the motion of a body sliding down a smooth Inclined Plane is similar to that of a body falling freely; the only difference is that $g \sin \alpha$ must be put instead of $g$ in the formulæ of Art. 26, so that the motion of the sliding body is slower than that of the body falling freely.

We shall in Chapter VII. consider the reason which justifies the *assumption* of the present Article.

28. Thus if $l$ be the length of an Inclined Plane, and $v$ the velocity acquired by a body in sliding down it from rest, we have $v^2 = 2gl \sin \alpha$ by equation (3) of Art. 26. Let $h$ be the height of the Plane; then $h = l \sin \alpha$; thus $v^2 = 2gh$.

*Hence the velocity acquired in sliding down a smooth Inclined Plane is the same as would be acquired in falling freely through a vertical space equal to the height of the Plane.*

29. *The time of falling from rest down a chord of a vertical circle drawn from the highest point is constant.*




Let $A$ be the highest point of a vertical circle, $AB$ a diameter, $AC$ any chord. Let $\alpha$ be the inclination of $AC$ to the horizon; then the angle $BAC = 90^0 - \alpha$, and therefore the angle $ABC = \alpha$.

Let $t$ be the time of falling down $AC$; then by Art. 27

$$AC = \frac{1}{2} t^2 g \sin \alpha.$$

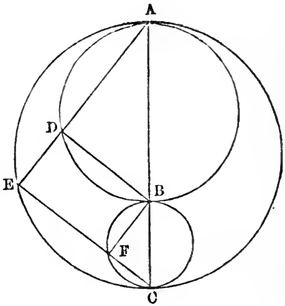
And $AC = AB \sin \alpha$; so that $AB \sin \alpha = \frac{1}{2} t^2 g \sin \alpha$;

therefore
$$t = \sqrt{\frac{2AB}{g}}.$$

That is, $t$ is equal to the time of falling freely down the vertical diameter $AB$. This establishes the proposition.

In the same manner we may shew that *the time of falling from rest down a chord passing through the lowest point is constant.*

30. *If two circles touch each other at their highest or lowest point, and a straight line be drawn through this point, the time of falling from rest down a straight line intercepted between the circumferences is constant.*




Let two circles touch each other at their highest point $A$. Through $A$ draw any straight line $ADE$, cutting the circumferences at $D$ and $E$. Let the vertical straight line through $A$ cut the circumferences at $B$ and $C$. On $BC$ as diameter describe a circle; join $EC$, cutting the circumference of this circle at $F$. Join $BF$ and $BD$.

The angles at $D$, $E$, and $F$ are right angles. Therefore $BF$ is parallel and equal to $DE$.

Hence the time from rest down $DE$ is the same as the time from rest down $BF$; and is therefore equal to the time from rest down $BC$, by Art. 29. Thus the time is the same for every such straight line as $DE$.

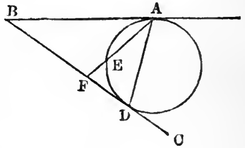
Similarly the proposition may be established when the circles touch at their lowest point.

31. The two preceding results will enable us to solve various problems with respect to *straight lines of quickest descent.* We will give some examples: we suppose in every example that the entire figure is in one vertical

plane, and the moving body is supposed to start from rest. The first six Examples depend on Art. 29, the rest on Art. 30.

Required the straight line of quickest descent:

(1) *From a given point to a given straight line.*




Let $A$ be the given point, $BC$ the given straight line, $AB$ a horizontal straight line through $A$. Draw a circle touching $AB$ at $A$ and also touching $BC$; let $D$ be the point of contact with $BC$: then $AD$ is the required straight line. For draw through $A$ any chord of the circle $AE$, and produce it to meet the straight line $BC$ at $F$. Then the time down $AD$ is equal to the time down $AE$, and is therefore *less* than the time down $AF$.

Since the two tangents $BA$ and $BD$ are equal, the point $D$ is determined simply by taking $BD$ down $BC$, equal to $BA$.

The demonstration of this will give sufficient aid for the next five cases.

(2) *From a given straight line to a given point.*

Let $A$ denote the given point; let a horizontal straight line through $A$ meet the given straight line at $B$; take $BD$ up the given straight line $= BA$: then $DA$ is the required straight line.

(3) *From a given point without a given circle to a given circle.*

Join the given point with the *lowest* point of the given circle: the part of the joining straight line which is outside the given circle is the straight line required.

For the geometrical part of this and the next three cases see *Appendix to Euclid*, No. 9.

(4) *From a given circle to a given point without it.*

Join the given point with the *highest* point of the given circle: the part of the joining straight line which is outside the given circle is the straight line required.

(5) *From a given point within a given circle to the circle.*

Join the given point with the *highest* point of the given circle: the part of the joining straight line produced which is between the point and the circle is the straight line required.

(6) *From a given circle to a given point within it.*

Join the given point with the *lowest* point of the given circle: the part of the straight line produced which is between the circle and the point is the straight line required.

(7) *From a given straight line without a given circle to the circle.*

Through $A$ the lowest point of the circle draw a straight line touching the circle, and meeting the given straight line at $B$; take $BC$ up the given straight line $=BA$, and join $AC$ meeting the circle at $D$: then $CD$ is the required straight line.

For it follows from (3) that whatever be the point on the straight line, the straight line produced must pass through the lowest point of the given circle. And then, by Art. 30, the point on the straight line must be the point of contact of a circle drawn to touch this straight line and also to touch the given circle at its lowest point.

(8) *From a given circle to a given straight line without the circle.*

Through $A$ the highest point of the circle draw a straight line touching the circle, and meeting the given straight line at $B$; take $BC$ down the given straight line $=BA$, and join $AC$ meeting the circle at $D$; then $DC$ is the required straight line.

The demonstration is like that in (7).

(9) *From a given circle to another given circle without it.*

Join the highest point of the first circle with the lowest point of the second circle; the part of this straight line which is between the two circles is the straight line required.

For it follows from (4) that whatever be the point on the second circle the straight line produced must pass through the highest point of the first circle. And then, by Art. 30, the point on the second circle must be the point of contact of a circle drawn to touch the first circle at its highest point, and also to touch the second circle.

The demonstration of the next two cases is similar to this.

(10) *From a given circle to another given circle within it.*

Join the lowest point of the first circle with the lowest point of the second circle: the part of the joining straight line which is between the two circles is the straight line required.

(11) *From a given circle within another given circle to the outer circle.*

Join the highest point of the first circle with the highest point of the second circle: the part of the joining straight line which is between the two circles is the straight line required.

32. In the preceding three Articles we have supposed for simplicity that the motion takes place in a *vertical* plane: but similar results will hold if the motion takes place down a fixed smooth Inclined Plane. If $\beta$ be the inclination of such a Plane to the horizon, then we shall merely have to put $g \sin \beta$ instead of $g$ in the investigation of Arts. 29 and 30. And in Arts. 29 and 30 we may put *sphere* instead of *circle.*

33. The following problem furnishes an interesting application of the formulæ of the present Chapter. A

person drops a stone into a well and after $n$ seconds hears it strike the water: find the depth of the surface of the water.

We neglect the resistance of the air. It appears from experiments that the velocity of sound is uniform and equal to about 1130 feet per second: we will denote this number by $u$.

Let $x$ be the number of feet in the depth of the surface; then the number of seconds taken by the stone to fall to the surface of the water is $\sqrt{\frac{2x}{g}}$, by Art. 26; and the number of seconds taken by the passage of the sound is $\frac{x}{u}$: therefore

$$\frac{x}{u} + \sqrt{\frac{2x}{g}} = n.$$

By solving this quadratic equation we obtain

$$2\sqrt{x} = -u\sqrt{\frac{2}{g}} \pm \sqrt{\left(\frac{2u^2}{g} + 4un\right)}.$$

The upper sign must be taken because $\sqrt{x}$ is by supposition a positive quantity. By squaring we obtain

$$4x = \frac{2u^2}{g} + \frac{2u^2}{g} + 4un - 4u\sqrt{\left(\frac{u^2}{g^2} + \frac{2un}{g}\right)};$$

therefore
$$x = \frac{u^2}{g} + un - u\sqrt{\left(\frac{u^2}{g^2} + \frac{2un}{g}\right)}$$
$$= u\left\{\frac{u}{g} + n - \sqrt{\left(\frac{u^2}{g^2} + \frac{2un}{g}\right)}\right\}.$$

It will be found that $\frac{u}{g} = 35$ nearly: thus we have very approximately

$$x = u\{35 + n - \sqrt{(1225 + 70n)}\}.$$

For example, if $n = 3$; then

$$x = u\{38 - \sqrt{1435}\} = u(38 - 37{\cdot}88) \text{ nearly}$$
$$= u \times {\cdot}12 = 135{\cdot}6.$$

34. In Art. 8 we have explained the change made in the expression of a given *velocity* by changing the units of time and space : we must now consider the change made in the expression of a given *acceleration*.

Let $f$ denote an acceleration when a second is taken as the unit of time, then the *same* acceleration will be denoted by $(60)^2 f$ when a minute is taken as the unit of time. For an acceleration is measured by the velocity communicated in a unit of time. In the present case $f$ is communicated in one second, therefore $2f$ in two seconds, ....and $60f$ in 60 seconds. But by Art. 8 a velocity which is denoted by $60f$ when a second is the unit of time will be denoted by $60 \times 60f$ when a minute is the unit of time. Hence $(60)^2 f$ is the measure of the acceleration referred to a minute as the unit of time.

In like manner if we wish to take a yard for the unit of length instead of a foot, as well as a minute for the unit of time instead of a second, the acceleration denoted by $f$ with the standard units will now be denoted by $\frac{(60)^2 f}{3}$.

Generally, let $f$ denote an acceleration when a second is the unit of time and a foot is the unit of length; then if we take $m$ seconds as the unit of time, and $n$ feet as the unit of length, the *same* acceleration will be denoted by $\frac{m^2}{n} f$.

## EXAMPLES. III.

The following examples all relate to uniformly accelerated motion :

1. A body has described 50 feet from rest in 2 seconds : find the velocity acquired.

2. A body has described 50 feet from rest in 2 seconds : find the time it will take to move over the next 150 feet.

3. A body moves over 63 feet in the fourth second. find the acceleration.

4. A body describes 72 feet while its velocity increases from 16 to 20 feet per second: find the whole time of motion, and the acceleration.

5. A body in passing over 9 feet has its velocity increased from 4 to 5 feet per second: find the whole space described from rest, and the acceleration.

6. Two bodies uniformly accelerated in passing over the same space have their velocities increased from $a$ to $b$, and from $u$ to $v$ respectively: compare the accelerations.

7. Find the numerical value of the acceleration when in half a second a velocity is produced which would carry a body over four feet in every quarter of a second.

8. A body moving from rest is observed to move over 80 feet and 112 feet respectively in two consecutive seconds: find the acceleration, and the time from rest.

9. A body moving from rest is observed to move over $a$ feet and $b$ feet respectively in two consecutive seconds: shew that the acceleration is $b-a$, and find the time from rest.

10. A body uniformly accelerated is found to be moving at the end of 10 seconds with a velocity which would carry it through 45 miles in the next hour: find the acceleration.

11. A body moving with uniform acceleration describes 20 feet in the half second which follows the first second of its motion: find the acceleration.

12. Two bodies are let fall from the same point at an interval of one second: find how many feet they will be apart at the end of five seconds from the fall of the first.

13. Two particles are let fall from two given heights: find the interval between their starting if they reach the ground at the same instant.

14. A body is let fall: find how many inches it moves over in the first half second of its motion: if it were

made to move uniformly during the next half second with the velocity then acquired, find over what space it would move.

15. A body slides down a smooth Inclined Plane of given height: shew that the time of its descent varies as the secant of the inclination of the Plane to the vertical.

16. A body falls to the ground; it describes $\frac{16}{25}$ of the whole space during the last second of the motion: find the whole time.

17. Find the position of a point on the circumference of a circle so that the time of descent down an Inclined Plane to the centre of the circle may be equal to the time of descent down an Inclined Plane to the lowest point of the circle.

18. Find a point in a vertical circle such that the time down a tangent at that point terminating in the vertical diameter produced may be equal to the time down the vertical diameter.

19. Find the measure of the force of gravity when half a second is taken as the unit of time.

20. Also when the unit of space is a metre, that is, about 3·28 feet.

21. Also when the unit of time is ten seconds, and the unit of space is a yard.

22. Also when the unit of time is a quarter of a second, and the unit of space is half a yard.

23. If $f$ be the measure of an acceleration when $m$ seconds is the unit of time, and $n$ feet the unit of length, find the measure of acceleration when a second and a foot are the units.

24. If $f$ be the measure of an acceleration when $m$ seconds is the unit of time, and $n$ feet the unit of length; find the measure of the acceleration when $\mu$ seconds is the unit of time, and $\nu$ feet the unit of length.

IV. *Motion in a straight line under the influence of a uniform force, with given initial velocity.*

35. In the preceding Chapter we confined ourselves to the case in which the body was supposed to have no velocity before the force began to operate; this supposition is usually expressed by saying that the body has no initial velocity. We shall now suppose that the body has an initial velocity, the direction of which coincides with the straight line in which the force acts.

36. *A body starts with the velocity* u, *and is acted on by a uniform force in the direction of this velocity during the time* t: *if* f *be the acceleration, and* v *the velocity of the body at the time* t, *then* $v=u+ft$.

For, by the definition of uniform force, in each unit of time the velocity $f$ is communicated to the body; and therefore in $t$ units of time the velocity $ft$ is communicated: therefore at the end of the time $t$ the velocity is $u+ft$.

37. *A body starts with the velocity* u, *and is acted on by a uniform force in the direction of the velocity during the time* t: *if* f *be the acceleration, and* s *the space described in the time* t, *then* $s=ut+\frac{1}{2}ft^2$.

Let the whole time $t$ be divided into $n$ equal intervals; denote each interval by $\tau$, so that $n\tau=t$. Then the velocity of the body at the end of the times

$$\tau,\ 2\tau,\ 3\tau,\ \ldots\ldots(n-1)\tau,\ n\tau$$

from starting is, by Art. 36, respectively

$$u+f\tau,\ u+2f\tau,\ u+3f\tau\ldots\ldots\ldots u+(n-1)f\tau,\ u+nf\tau.$$

Let $s_1$ denote the space which the body would describe if it moved during each interval $\tau$ with the velocity which it has at the *beginning* of the interval; and let $s_2$ denote

the space which the body would describe if it moved during each interval $\tau$ with the velocity which it has at the *end* of the interval. Then

$$s_1 = u\tau + \{u + f\tau\}\tau + \{u + 2f\tau\}\tau + \ldots\ldots + \{u + (n-1)f\tau\}\tau,$$

$$s_2 = \{u + f\tau\}\tau + \{u + 2f\tau\}\tau + \ldots$$
$$\ldots + \{u + (n-1)f\tau\}\tau + \{u + nf\tau\}\tau;$$

that is,

$$s_1 = un\tau + f\tau^2\{1 + 2 + 3 + \ldots\ldots + (n-1)\},$$
$$s_2 = un\tau + f\tau^2\{1 + 2 + 3 + \ldots\ldots + (n-1) + n\}.$$

Hence, by the theory of Arithmetical Progression in Algebra, we have

$$s_1 = un\tau + f\tau^2\frac{(n-1)n}{2} = ut + \frac{ft^2}{2}\left(1 - \frac{1}{n}\right),$$

$$s_2 = un\tau + f\tau^2\frac{n(n+1)}{2} = ut + \frac{ft^2}{2}\left(1 + \frac{1}{n}\right).$$

Now $s$, the space actually described, must lie between $s_1$ and $s_2$; but by making $n$ large enough we can make $\frac{1}{n}$ as small as we please; so that we can make $s_1$ and $s_2$ differ from $ut + \frac{1}{2}ft^2$ by less than any assigned quantity. Hence $s = ut + \frac{1}{2}ft^2$.

38. The result just obtained has been deduced by an independent investigation founded on first principles; if we are allowed to assume the result obtained in Art. 21 we may put the investigation more briefly as follows:

If the body at a certain instant is moving with a certain velocity, its subsequent motion will be the same, however we suppose that velocity to have been acquired. Let us suppose that the velocity $u$ was generated by the action of the force, of which the acceleration is $f$, during the

time $t'$; and let the body have moved from rest through the space $s'$ during this time. Then we have, by Art. 21,

$$u = ft',$$

$$s' = \frac{1}{2} ft'^2,$$

$$s' + s = \frac{1}{2} f(t' + t)^2 = \frac{1}{2} ft'^2 + ft't + \frac{1}{2} ft^2;$$

therefore $$s = ft't + \frac{1}{2} ft^2 = ut + \frac{1}{2} ft^2.$$

39. The result of Art. 37 is sometimes obtained in the following way:

If no force acted on the body the space described in the time $t$ would be $ut$, by Art. 5. If there were no initial velocity the space described in the time $t$ under the influence of the force would be $\frac{1}{2} ft^2$. Now if the body start with the velocity $u$, and be also acted on by the force, the space actually described must be the sum of these two spaces; because by the nature of uniform force the *velocity at any instant is exactly the sum of what it would be in the two supposed cases.*

40. Hence we have the following results when a body starts with a given velocity and is acted on by a uniform force in the direction of this velocity:

Let $f$ be the acceleration, $u$ the initial velocity, $v$ the velocity at the end of the time $t$, and $s$ the space described; then

$$v = u + ft \quad \text{.......................(1).}$$

$$s = ut + \frac{1}{2} ft^2 \quad \text{....................(2).}$$

From (1) and (2) we have

$$v^2 = u^2 + 2uft + f^2t^2 = u^2 + 2f\left(ut + \frac{1}{2} ft^2\right);$$

thus $$v^2 = u^2 + 2fs \quad \text{.................(3).}$$

41. The student must observe that *during* the motion which we consider in Art. 37 the *only* force acting is that of which the acceleration is $f$. The body starts with the velocity $u$, and this must have been generated by some force, which may have been *sudden*, as a blow or an explosion is usually considered to be, or may have been *gradual* like the force of gravity. But we are only concerned with what takes place *after* this velocity $u$ has been generated, and so during the motion which we consider no force acts except that of which the acceleration is $f$.

42. Hitherto we have supposed the direction of the force to be the *same* as that of the initial velocity; we will now consider the case in which the direction of the force is *opposite* to that of the initial velocity. It will be sufficient to state the results, which can be obtained as in Arts. 36, 37, 38, and 40.

Let $f$ be the acceleration, $u$ the initial velocity, $v$ the velocity at the end of the time $t$, and $s$ the space described, the force and the initial velocity being in *opposite* directions; then

$$v = u - ft \qquad (1),$$

$$s = ut - \frac{1}{2}ft^2 \qquad (2),$$

$$v^2 = u^2 - 2fs \qquad (3).$$

These formulæ will present some interesting consequences; the student will obtain an illustration of the interpretation ascribed in Algebra to the negative sign.

As long as $ft$ is less than $u$ we see from (1) that $v$ is positive, so that the body is moving in the direction in which it started. When $ft - u = 0$, that is when $t = \frac{u}{f}$, we have $v = 0$, so that the body is for an instant at rest. When $t$ is greater than $\frac{u}{f}$ the value of $v$ is *negative;* that is, the body is moving in the direction *opposite* to that in which it started. Thus we see that the body continues to move in the direction in which it started, until by the

operation of the force, which acts in the opposite direction, all its velocity is destroyed; after this the force generates a new velocity in the body in the direction of the force, that is, in the direction opposite to that of the original velocity.

From (2) when $t=\frac{u}{f}$ we have $s=\frac{u^2}{f}-\frac{1}{2}\frac{u^2}{f}=\frac{u^2}{2f}$; this gives the whole space described by the body while moving in the direction in which it started. This value of $s$ may also be obtained from (3) by putting $v=0$; for then we have $u^2-2fs=0$.

From (2) we have $s=0$ when $ut-\frac{1}{2}ft^2=0$, that is when $t=0$ and when $t=\frac{2u}{f}$. The value $t=0$ corresponds to the instant of starting; the other value of $t$ must correspond to the instant when the body in its backward course reaches the starting point again. Thus the time taken in moving backwards from the turning point to the starting point is $\frac{2u}{f}-\frac{u}{f}$, or $\frac{u}{f}$, which is equal to the time taken in moving forwards from the starting point to the turning point. Put $t=\frac{2u}{f}$ in (1), then we get $v=u-2u=-u$; so that at this instant the velocity of the body is the same *numerically* as it was at starting, but in the *opposite* direction. Equation (3) shews that the velocity at any point of the forward course is *numerically* the same as at the same point of the backward course. When $t$ is greater than $\frac{2u}{f}$ the value of $s$ becomes *negative*, indicating that the body is now on the side of the starting point *opposite* to that on which it was while $t$ changed from 0 to $\frac{2u}{f}$.

It will be useful to remember these two results: the original velocity $u$ is destroyed in the time $\frac{u}{f}$, and the space described in that time is $\frac{u^2}{2f}$.

43. The most important application of the preceding Article is to the case of gravity. If a body be projected vertically upwards with a velocity $u$ it rises for a time $\frac{u}{g}$, reaches the height $\frac{u^2}{2g}$, falls to the ground in the same time as it took to rise, and strikes the ground with the velocity $u$ downwards.

## EXAMPLES. IV.

1. A stone is thrown vertically upwards with a velocity $3g$: find at what times its height will be $4g$, and find its velocity at these times.

2. A body is projected vertically upwards with a velocity which will carry it to a height $2g$: find after what interval the body will be descending with the velocity $g$.

3. A body moves over 20 feet in the first second of time during which it is observed, over 84 feet in the third second, and over 148 feet in the fifth second: determine whether this is consistent with the supposition of uniform acceleration.

4. A particle uniformly accelerated describes 108 feet and 140 feet in the fifth and seventh seconds of its motion respectively: find the initial velocity and the numerical measure of the acceleration.

5. A body starts with a certain velocity and is uniformly accelerated: shew that the space described in any time is equal to that which would be described in the same time with a uniform velocity equal to half the sum of the velocities at the beginning and at the end of the time.

6. A bullet shot upwards from a gun passes a certain point at the rate of 400 feet per second: find when the bullet will be at a point 1600 feet higher.

7. A body is dropped from a given height and at the same instant another body is started upwards, and they meet half way: find the initial velocity of the latter body.

8. At the same instant one body is dropped from a given height, and another body is started vertically upwards from the ground with just sufficient velocity to attain that height: compare the time they take before they meet with the time in which the first would have fallen to the ground.

9. A smooth Plane is inclined at an angle of $30^0$ to the horizon; a body is started up the Plane with the velocity $g$: find the time it takes to describe a space $g$.

10. A smooth Plane is inclined at an angle of $30^0$ to the horizon; a body is started up the Plane with the velocity $5g$: find when it is distant $9g$ from the starting point.

11. A body is thrown vertically upwards, and the time between its leaving a given point and returning to it again is observed: find the initial velocity.

12. A particle is moving under the action of a uniform force, the acceleration of which is $f$: if $p$ be the arithmetic mean of the first and last velocities in passing over any portion $s$ of the path, and $q$ the velocity generated, shew that $pq=fs$.

13. Two small heavy rings capable of sliding along a smooth straight wire of given length inclined to the horizon are started from the two extremities of the wire each with the velocity due to their vertical distance: find the time after which they will meet, and shew that the space described by each is independent of the inclination of the wire.

14. A body begins to move with the velocity $u$, and at equal intervals of time an additional velocity $v$ is communicated to it in the same direction: find the space described in $n$ such intervals. Hence deduce the space described from rest under the action of a force constant in magnitude and direction.

V. *Second Law of Motion. Motion under the influence of a uniform force in a fixed direction, but not in a straight line. Projectiles.*

44. We are still confining ourselves to the case of a uniform force in a fixed direction; but the body will now be supposed to start with a velocity which is not in the same direction as the force: it will appear that a body under such circumstances will not describe a straight line but a certain curve called a *parabola.*

It is necessary at this stage to introduce the *Second Law of Motion.*

45. *Second Law of Motion. Change of motion is proportional to the acting force, and takes place in the direction of the straight line in which the force acts.*

So long as we keep to the same force and the same body change of motion is measured by *change of velocity;* the law then asserts that any force will communicate velocity in the direction in which the force acts: and it is implied that the amount of the velocity so communicated does not depend on the amount or the direction of the velocity which may have been already communicated to the body. It will appear hereafter that the law contains *more* than this: see Art. 84.

For the reason explained in Art. 10 we ought to suppose the Second Law to relate to the motion of a *particle.*

46. In confirmation of the truth of the Second Law of Motion it is usual to adduce the following experiment: if a stone be dropped from the top of the mast of a ship in motion the stone will fall at the foot of the mast notwithstanding the motion of the ship. The stone does not fall in a straight line; it starts with a certain horizontal velocity, namely, the same as that of the ship, and gravity acts on it in a vertical direction. The fact that the stone falls at the foot of the mast shews that the vertical force

of gravity makes no change in the *horizontal* velocity with which the stone started; so that the *vertical* force can only have communicated a *vertical* velocity. If the time of the descent of the stone were observed, and found to be the same as of a stone falling from rest through the same space, the confirmation of the truth of the Second Law of Motion would be much more decisive.

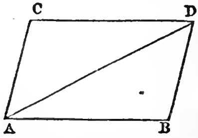
However it is obvious that few persons can perform this experiment: but most persons can observe the fact that the motion of a steamer or of a railway train will not affect the circumstances of the fall of bodies through small heights.

As we have already indicated in Art. 12, the best evidence of the truth of the Laws of Motion is the agreement of results deduced from these Laws with observed phenomena, especially those furnished by Astronomy.

47. Newton gives the following as one of the Corollaries to his Laws of Motion:

A body acted on by two forces will describe the diagonal of a parallelogram in the time in which it would describe the sides under the influence of the forces singly.

The following is the substance of Newton's exposition of this statement.




Suppose that a body, in a given time, under the influence of a single force $M$, which acted at $A$, would move with uniform velocity from $A$ to $B$; and suppose that the body in the same time under the influence of another single force $N$, which acted at $A$, would move with uniform velocity from $A$ to $C$; complete the parallelogram $ABCD$: then if both forces act simultaneously at $A$ the body will move uniformly in the given time from $A$ to $D$.

For since the force $N$ acts along the straight line $AC$, which is parallel to $BD$, this force, by the Second Law of Motion, will not change the velocity of approach towards the straight line $BD$, which is produced by the other force.

Thus the body will reach the straight line $BD$ in the same time whether the force $N$ act or not: and so at the end of the given time will be found somewhere in the straight line $BD$. By the same reasoning it follows that the body at the end of the given time will be found somewhere in the straight line $CD$. Therefore the body will be at $D$.

The body must move in a *straight* line from $A$ to $D$, by the First Law of Motion.

48. Thus it appears that, according to Newton's view, the Second Law of Motion tells us that when forces act simultaneously on a body each force communicates in a given time the same velocity as if it acted singly on the body originally at rest; and then by the Corollary we learn how to compound the velocities thus generated into a single velocity.

It will be seen that Newton supposes in his exposition that the two forces act *instantaneously;* that is, they are of the kind which we naturally suppose a *blow* to be, and communicate velocity by *sudden* action, not by *continuous* action.

49. The principle contained in Art. 47 is called the *Parallelogram of Velocities*, and is usually enunciated thus: *if a body have communicated to it simultaneously two velocities which are represented in magnitude and direction by two straight lines drawn from a point, then the resultant velocity will be represented in magnitude and direction by the diagonal, drawn from that point, of the parallelogram constructed on the two straight lines as adjacent sides.*

This principle gives rise to applications similar to that deduced from the Parallelogram of Forces in Statics. We may use the principle either to compound two velocities into one, or to resolve one velocity into two.

50. Thus if velocities $u$ and $v$ be simultaneously communicated to a body in directions which include an angle $\alpha$, the resultant velocity is $\sqrt{(u^2+v^2+2uv\cos\alpha)}$. Let $\beta$ be the angle between the direction of the velocity $u$ and that of

the resultant velocity, and $\gamma$ the angle between the direction of the velocity $v$ and that of the resultant velocity; then $\beta+\gamma=a$, and

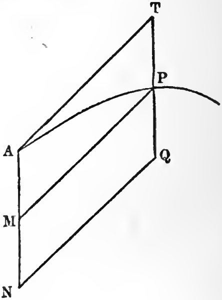
$$\frac{\sin\beta}{\sin\gamma}=\frac{v}{u}.$$

In the special case in which $a=90^0$, the resultant velocity is $\sqrt{(u^2+v^2)}$; also

$$\sin\beta=\frac{v}{\sqrt{(u^2+v^2)}},\quad \sin\gamma=\frac{u}{\sqrt{(u^2+v^2)}}.$$

See *Statics*, Art. 30.

51. *A body projected in any direction not vertical and acted on by gravity will describe a parabola.*




Let a body be projected from the point $A$ in any direction which is not vertical; let $AT$ be the space which would be described by the body in the time $t$ if the force of gravity did not act. Draw $AM$ vertically downwards, equal to the space through which a body would fall from rest, in the time $t$, under the action of gravity. Complete the parallelogram $ATPM$. Then $P$, the corner opposite to $A$, will be the place of the body at the end of the time $t$.

For, by the Second Law of Motion, gravity will communicate the same vertical velocity to the body as it would if the body had not received any other velocity. Thus at any instant there will be the same vertical velocity as if there had been no velocity parallel to $AT$, and the same velocity parallel to $AT$ as if there had been no vertical velocity. Therefore the spaces described parallel to $AT$ and $AM$ respectively will be the same as if each alone had been described. Thus $P$ will be the place of the body at the end of the time $t$.

Let $u$ be the velocity with which the body is projected at $A$; then $AT$ or $PM = ut$; also $AM = \frac{1}{2} gt^2$, therefore

$$PM^2 = \frac{2u^2}{g} AM.$$

Thus $PM^2$ bears a constant ratio to $AM$, and therefore by Conic Sections the path of the body is a parabola, having its axis vertical and $AT$ for a tangent. And $\frac{u^2}{2g}$ is the distance of $A$ from the focus of the parabola, and also from the directrix.

52. Produce $TP$ to $Q$, so that $PQ = PT$, and produce $AM$ to $N$ so that $MN = AM$. Then

$$\frac{AT}{AN} = \frac{ut}{gt^2} = \frac{u}{gt}.$$

Now at the end of the time $t$ the body has its original velocity parallel to $AT$, and also the vertical velocity $gt$ downwards which has been communicated to it by gravity; these velocities are proportional to $AT$ and $AN$, and in these directions. Hence the resultant velocity at $P$ is parallel to $AQ$, by Art. 49. Thus if this direction be drawn at $P$, and be produced, it will meet the vertical straight line through $A$ at a point whose distance from $A$ is equal to $PQ$, that is to $TP$. The direction thus determined for the velocity at $P$ is, by Conic Sections, that of the *tangent* at $P$, which might have been anticipated.

53. A body projected in any manner and acted on by gravity is called a *Projectile;* thus we have shewn that the path of a Projectile is in general a parabola, and is a straight line in the particular case in which the body is projected vertically upwards or downwards. The student must observe that *during* the motion which we consider in Art. 51 the only *force* acting is that of gravity; see Art. 41.

54. *The velocity of a projectile at any point of its path is that which would be acquired in falling from the directrix to the point.*

For we have seen in Art. 51 that the distance of the directrix from the point of projection is $\frac{u^2}{2g}$, where $u$ is the velocity of projection; and $\frac{u^2}{2g}$ is equal to the vertical space through which a body must fall from rest under the action of gravity in order to acquire the velocity $u$. Now any point of the parabolic path may be regarded as the point of projection, and the velocity at that point as the velocity of projection. Thus the required result is obtained.

55. The preceding result may also be obtained thus:

Suppose the direction of projection to make an angle $a$ with the horizon; resolve the velocity of projection $u$ into $u \cos a$ in the horizontal direction and $u \sin a$ in the vertical direction. Then at the end of the time $t$ the horizontal velocity is still $u \cos a$, and the vertical velocity is $u \sin a - gt$.

Let $v$ denote the resultant velocity; then, by Art. 50,

$$v^2 = (u \cos a)^2 + (u \sin a - gt)^2$$
$$= u^2 - 2gtu \sin a + g^2t^2 = u^2 - 2g\left(tu \sin a - \frac{1}{2}gt^2\right).$$

Now $tu \sin a - \frac{1}{2}gt^2$ is the vertical height of the body above the horizontal plane through the starting point by Art. 43; we will denote this by $y$: thus $v^2 = u^2 - 2gy$.

Let $h$ denote the distance of the directrix from the starting point, so that $h = \frac{u^2}{2g}$; thus $v^2 = 2g(h - y)$.

This shews that the velocity is that which would be acquired in falling from rest through the space $h - y$, that is, in falling from the directrix to the point considered.

56. *To determine the position of the focus of the parabola described by a projectile.*

Let $u$ be the velocity of projection, and $a$ the angle which the direction of projection makes with the horizon.

The distance of the focus from the point of projection is $\frac{u^2}{2g}$ by Art. 51. By the nature of the parabola the tangent at any point makes equal angles with the focal distance of that point and the diameter at the point. Hence the straight line from the point of projection to the focus makes an angle $2(90^0 - a)$ with the vertical, and therefore an angle $2a - 90^0$ with the horizon. Thus the situation of the focus is determined.

The height of the focus above the horizontal plane through the point of projection is $\frac{u^2}{2g}\sin(2a - 90^0)$, that is $-\frac{u^2}{2g}\cos 2a$. Thus the focus is *below* the horizontal plane through the point of projection if $2a$ is less than $90^0$, and *above* it if $2a$ is greater than $90^0$.

If a perpendicular be drawn from the focus on the horizontal plane through the point of projection, the distance of the foot of the perpendicular from the point of projection is $\frac{u^2}{2g}\cos(2a - 90^0)$, that is $\frac{u^2}{2g}\sin 2a$.

57. *To find the time in which a projectile reaches its greatest height, and the greatest height.*

Let $u$ be the velocity of projection, $a$ the angle which the direction of projection makes with the horizon; then at the end of the time $t$ the vertical velocity is $u\sin a - gt$. Now at the instant of reaching the greatest height the vertical velocity vanishes, so that we have $u\sin a - gt = 0$; therefore $t = \frac{u\sin a}{g}$.

By Art. 55 the height of the projectile at the time $t$ above the horizontal plane through the point of projection is $tu\sin a - \frac{1}{2}gt^2$; substitute the value of $t$ just found: thus the greatest height is $\frac{u^2\sin^2 a}{g} - \frac{1}{2}\frac{u^2\sin^2 a}{g}$, that is $\frac{u^2\sin^2 a}{2g}$. Compare Art. 43.

The position of the highest point may be easily determined in the manner of Art. 56.

58. *To determine the Latus Rectum of the parabola described by a projectile.*

Let $u$ be the velocity of projection, $\alpha$ the angle which the direction of projection makes with the horizon.

At the highest point the velocity is entirely *horizontal*, so that it is parallel to the directrix; and thus the highest point is the *vertex* of the parabola. The velocity at the highest point is $u \cos \alpha$. By Art. 54 this velocity would be acquired in falling from the directrix; therefore the distance of the vertex from the directrix is $\frac{u^2\cos^2\alpha}{2g}$. The latus rectum is equal to four times this distance, so that it is $\frac{2u^2\cos^2\alpha}{g}$.

59. The interval between the projection of a projectile and its return to the horizontal plane through the point of projection is called the *time of flight.* The distance from the point of projection of the point at which the body meets the horizontal plane is called the *range on the horizontal plane through the point of projection.*

60. *To find the time of flight of a projectile.*

Let $u$ be the velocity of projection, $\alpha$ the angle which the direction of projection makes with the horizon.

The height of the body at the time $t$ is $tu \sin \alpha - \frac{1}{2} gt^2$.

This vanishes when $t=0$ and when $t=\frac{2u \sin \alpha}{g}$. The value $t=0$ corresponds to the instant of starting; the other value of $t$ must correspond to the instant when the body again reaches the horizontal plane through the point of projection. Thus by Art. 57 we see that the time which the projectile takes in descending from the highest point to the horizontal plane through the point of projection is $\frac{2u \sin \alpha}{g} - \frac{u \sin \alpha}{g}$, that is $\frac{u \sin \alpha}{g}$; so that the time of descent is equal to the time of ascent.

61. On reaching the ground the vertical velocity is $u \sin a - 2g \dfrac{u \sin a}{g}$, that is $-u \sin a$; thus it is numerically the same as at starting, but in the opposite direction. The horizontal velocity is the same at the two points.

And generally at the two points which are in the same horizontal plane the whole velocities are the same by Art. 54; and the horizontal velocities are the same: hence the vertical velocities are numerically the same, but must be in opposite directions.

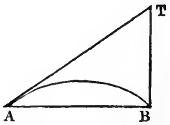
62. *To find the range on the horizontal plane through the point of projection.*

It is shewn in Art. 60 that the time of flight is $\dfrac{2u \sin a}{g}$; and the horizontal velocity is $u \cos a$: hence the horizontal space described is $\dfrac{2u \sin a}{g} \times u \cos a$. This may be put in the form $\dfrac{u^2}{g} \sin 2a$.

It is often useful to observe that it is $\dfrac{2}{g}\, u \sin a \,.\, u \cos a$, that is $\dfrac{2}{g} \times$ vertical velocity at starting $\times$ horizontal velocity.

63. The time of flight and the range may also be investigated thus:

Let $A$ be the point of projection, $AT$ the direction of projection, $AB$ the range on the horizontal plane through $A$, and $TB$ vertical. Let $u$ be the velocity of projection, $a$ the angle $TAB$, $t$ the time of flight.




Then $AT = ut$, $\quad TB = \frac{1}{2} g t^2$; hence

$$\sin a = \frac{TB}{AT} = \frac{\frac{1}{2} g t^2}{ut} = \frac{gt}{2u}; \quad \text{therefore } t = \frac{2u \sin a}{g}.$$

And $AB = AT \cos a = tu \cos a = \dfrac{2u^2 \sin a \cos a}{g}$.

64. *To determine the inclination to the horizon of the direction of motion of a projectile at any instant.*

By Art. 55 the vertical and horizontal velocities at the end of any time $t$ are respectively

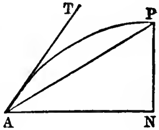
$$u \sin a - gt \text{ and } u \cos a.$$

Thus if $v$ be the resultant velocity, and $\phi$ the angle which its direction makes with the horizon,

$$v \sin \phi = u \sin a - gt, \quad v \cos \phi = u \cos a;$$

therefore $$\tan \phi = \frac{u \sin a - gt}{u \cos a}.$$

65. *An inclined plane passes through the point of projection of a projectile and is at right angles to the plane of motion: to find the time of flight, the greatest distance from the plane, and the range on the plane.*




Let $AP$, the inclined plane, make an angle $\beta$ with the horizon $AN$; let $AT$, the direction of projection, make an angle $a$ with the horizon; let $u$ be the velocity of projection.

Resolve the initial velocity along the plane and at right angles to it; the latter part is $u \sin (a - \beta)$. Resolve the acceleration $g$ parallel to the plane and at right angles to it; the latter part is $g \cos \beta$.

The motion in the direction at right angles to the plane is independent of the motion parallel to the plane. Hence as in Arts. 43 and 57 the body reaches its greatest distance from the plane at the end of the time $\frac{u \sin (a - \beta)}{g \cos \beta}$; and it takes the same time to move from this point to the plane, so that the time of flight is $\frac{2u \sin (a - \beta)}{g \cos \beta}$.

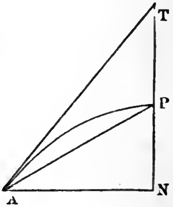
And, as in Arts. 43 and 57, the greatest distance from the plane is $\frac{u^2 \sin^2 (a - \beta)}{2g \cos \beta}$.

Let $P$ be the point where the body meets the inclined plane: draw $PN$ perpendicular to the horizon. Thus $AN = AP \cos\beta$.

But $AN$ is the horizontal space described in the time $\frac{2u\sin(\alpha-\beta)}{g\cos\beta}$; therefore $AN = \frac{u\cos\alpha\, 2u\sin(\alpha-\beta)}{g\cos\beta}$;

therefore
$$AP = AN\sec\beta = \frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}.$$

The preceding result may also be obtained thus:




Let $A$ be the point of projection, $P$ the point where the body meets the inclined plane. Draw a vertical line meeting the direction of projection at $T$ and the horizon at $N$.

Then, with the same notation as before,

$$AT = ut, \quad TP = \frac{1}{2}gt^2;$$

also
$$\frac{TP}{AT} = \frac{\sin TAP}{\sin APT} = \frac{\sin TAP}{\cos PAN}:$$

thus
$$\frac{\frac{1}{2}gt^2}{ut} = \frac{\sin(\alpha-\beta)}{\cos\beta}, \text{ whence } t = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}.$$

Also $AN = AT\cos TAN = tu\cos\alpha = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos\beta}$,

$$AP = AN\sec\beta = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}.$$

66. The theory of projectiles which has now been given is of no practical use, because it is found by experiment that the resistance of the air exercises a very powerful influence on the motion of a body, especially when the velocity is large. On this account the actual path of a cannon ball is not a parabola, and the range and time of flight are quite different from the values determined

above. The following is an example: a ball of certain size and weight being projected at an angle of $45^0$, with a velocity of 1000 feet per second, it is found that on taking the resistance of the air into account the range is about 5000 feet, instead of being $\frac{(1000)^2}{32}$.

The discussion of the motion of a projectile, taking into account the resistance of the air, is however far too difficult to find a place in this book.

## EXAMPLES. V.

1. Velocities of 5 feet and 12 feet per second in directions at right angles to each other are simultaneously communicated to a body: determine the resultant velocity.

2. A body is projected with the velocity $3g$ at an inclination of $75^0$ to the horizon: determine the range.

3. If at the highest point of the path of a projectile the velocity be altered without altering the direction of motion, will the time of reaching the horizontal plane which passes through the point of projection be altered?

4. From the highest point of the path of a projectile another body is projected horizontally with a velocity equal to the original vertical velocity of the first body: shew that the focus of the path described by the second body is in the horizontal plane which passes through the point of projection of the first body.

5. A ship is moving with a velocity $u$, a cannon ball is shot from a cannon which makes an angle $\alpha$ with the horizon, with powder which would give a velocity $v$ to the ball if the cannon were at rest: find the range supposing the ship and the ball to move in the same vertical plane.

6. Two bodies are projected simultaneously from the same point, with different velocities and in different directions in the same plane: find their distance apart at the end of a given time.

7. Determine how long a particle takes in moving from the point of projection to the further end of the latus rectum.

8. A body slides down a smooth Inclined Plane: shew that the distance between the foot of the Inclined Plane and the focus of the parabola which the particle describes after leaving the Plane is equal to the height of the Plane.

9. Two parabolic paths have a common focus and their axes in the same straight line: shew that if tangents be drawn to the two paths from any point in their common axis the velocities at the points of contact are equal.

10. If two projectiles have the same initial velocity and the same horizontal range, the foci of their paths are at equal distances from the horizontal plane, which passes through the point of projection.

11. A heavy particle is projected from a point with a given velocity, and in a given direction: find its distance from the point of projection at the end of a given time.

12. A number of particles are projected simultaneously from a fixed point in one plane, so that their least velocity is constant: shew that all of them will be found at any the same instant on the same vertical line.

13. A body is projected with a given velocity and in a given direction: determine the velocity with which another must be projected vertically so that the two may reach the ground at the same instant.

14. A ball fired with velocity $u$ at an inclination $\alpha$ to the horizon just clears a vertical wall which subtends an angle $\beta$ at the point of projection: determine the instant at which the ball just clears the wall.

15. In the preceding Example determine the horizontal distance between the foot of the wall and the point where the ball strikes the ground.

16. If one body fall down an Inclined Plane, and another be projected from the starting point horizontally along the Plane, find the distance between the two bodies when the first has descended through a given space.

## VI. *Projectiles continued.*

67. Although, as we have stated at the end of the preceding Chapter, the theory of projectiles is of no use in practice, yet it deserves careful study on account of the valuable illustration which it affords of the principles of Dynamics; and a thorough knowledge of the elementary principles is the true foundation for those higher investigations which apply to the phenomena actually presented by nature. A very large number of deductions and problems may be given which serve to impress the methods and results of the preceding Chapter on the memory: some of these Examples we will now discuss.

68. We have seen in Art. 62 that the range on the horizontal plane through the point of projection is $\frac{u^2}{g}\sin 2a$. Hence we deduce the following results:

The greatest range for a given velocity of projection is found by supposing $2a=90^0$, that is $a=45^0$: this greatest range is $\frac{u^2}{g}$.

Suppose the range to be given; denote it by $c$: then $\frac{u^2}{g}\sin 2a=c$, thus if either $a$ or $u$ is also given we may find the other.

Since $u^2=\frac{cg}{\sin 2a}$, the least value of $u$ is when $\sin 2a$ is greatest, that is when $a=45^0$.

Thus when $a=45^0$ we have the greatest range corresponding to a given velocity, and also the least velocity corresponding to a given range.

Again, suppose $c$ and $u$ given, and $a$ to be found; we have $\sin 2a=\frac{cg}{u^2}$: it is known by Trigonometry that if $cg$

is less than $u^2$ there are two values of $2a$ between 0 and $180^0$ which satisfy this equation, and one value is the *supplement* of the other. Hence there are two values of $a$ between 0 and $90^0$, and one value is the *complement* of the other.

69. We have seen in Art. 65 that the range on a plane inclined at an angle $\beta$ to the horizon which passes through the point of projection and is at right angles to the plane of motion is $\frac{2u^2 \cos a \sin (a-\beta)}{g \cos^2 \beta}$. We shall now investigate for what angle of projection this range is greatest, the velocity being given.

We have to investigate for what value of $a$ the expression $\cos a \sin (a-\beta)$ has its greatest value. Now we know by Trigonometry that

$$2 \cos a \sin (a-\beta) = \sin (2a-\beta) - \sin \beta;$$

hence the greatest value is when $2a-\beta=90^0$, that is when

$$a = \frac{1}{2}(\beta + 90^0);$$

the greatest range is

$$\frac{u^2 (1-\sin \beta)}{g \cos^2 \beta}, \text{ that is } \frac{u^2}{g(1+\sin \beta)}.$$

Suppose the range to be given; denote it by $c$: then

$$\frac{2u^2 \cos a \sin (a-\beta)}{g \cos^2 \beta} = c.$$

Hence the least value of $u$ is when $\cos a \sin (a-\beta)$ is greatest, that is, as before, when $a = \frac{1}{2}(\beta + 90^0)$.

Thus, for this value of $a$ we have the greatest range corresponding to a given velocity, and also the least velocity corresponding to a given range.

Suppose the range and the velocity of projection given, and that we have to find the angle of projection; then since

$$2 \cos \alpha \sin (\alpha - \beta) = \frac{gc \cos^2 \beta}{u^2},$$

we have $$\sin (2\alpha - \beta) = \sin \beta + \frac{gc \cos^2 \beta}{u^2}.$$

Hence we have in general two values of $2\alpha - \beta$ between 0 and $180^0$, and one value is the supplement of the other. Suppose one of these values is $\gamma$, then the other is $180^0 - \gamma$; from the former we obtain $\alpha = \frac{1}{2}(\beta + \gamma)$, and from the latter $\alpha = 90^0 + \frac{1}{2}(\beta - \gamma)$. The sum of these values of $\alpha$ is $90^0 + \beta$, that is twice the angle of projection which gives the greatest range corresponding to a given velocity: hence the two directions which correspond to a given range are equally inclined to that which corresponds to the greatest range, but on opposite sides of it.

70. *To determine the direction in which a body must be projected from a given point with a given velocity so as to hit a given point.*

Let $A$ denote the point of projection, $B$ the other given point. The velocity at $A$ is known; and therefore the distance of $A$ from the directrix is known by Art. 54, so that the position of the directrix is known. Then since $B$ is a given point, the distance of $B$ from the directrix is also known.

Now the distance of any point in the parabola from the focus is equal to the distance of that point from the directrix: hence the distances of $A$ and $B$ from the focus are known.

Describe a circle with $A$ as a centre, and radius equal to the known distance of the focus from $A$; describe another circle with $B$ as centre, and radius equal to the known distance of the focus from $B$. The focus of the

parabola will be at the intersection of the circles; and as the directrix is also known, the parabola is determined.

If the circles do not meet the problem has no solution; if they touch there is one solution; if they cut, since either point of intersection may be taken, there are two solutions.

71. Let $u$ be the velocity of projection, $\alpha$ the angle which the direction of projection makes with the horizon; and let $y$ be the height of the projectile at the time $t$ above the horizontal plane through the point of projection. Then as in Art. 55,

$$y = tu \sin \alpha - \frac{1}{2} g t^2 \quad \text{.................(1).}$$

Suppose a perpendicular to be drawn from the projectile at the end of the time $t$ on the horizontal plane through the point of projection; and let $x$ be the distance of the foot of the perpendicular from the starting point; then

$$x = tu \cos \alpha \quad \text{.................(2).}$$

From (2) we have $t = \dfrac{x}{u \cos \alpha}$; substitute in (1), thus

$$y = x \tan \alpha - \frac{g x^2}{2u^2 \cos^2 \alpha} \quad \text{..............(3).}$$

These equations are often useful in solving problems respecting projectiles.

72. We may apply equation (3) of the preceding Article to give another mode of solving the problem in Art. 70.

For since the point to be hit is given, the values of $x$ and $y$ will be known; we may then by solving the quadratic equation determine $\tan \alpha$. For we have

$$y = x \tan \alpha - \frac{g x^2}{2u^2}(1 + \tan^2 \alpha),$$

or

$$\tan^2 \alpha - \frac{2u^2}{gx} \tan \alpha + 1 + \frac{2yu^2}{gx^2} = 0.$$

Hence, by solving the quadratic equation,

$$\tan a = \frac{u^2}{gx} \pm \sqrt{\left(\frac{u^4}{g^2x^2} - 1 - \frac{2yu^2}{gx^2}\right)}.$$

Hence we see that $\tan a$ has two values, or one, or none, according as the quantity under the radical sign is positive, zero, or negative, that is, according as $\frac{u^4}{g^2}$ is greater than, equal to, or less than $x^2 + \frac{2yu^2}{g}$.

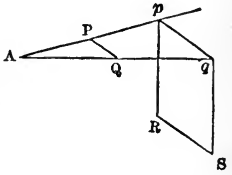
73. *If particles are projected from the same point, at the same instant, with the same velocity, in different directions, they will all at any future instant be on the surface of a sphere.*

Let $u$ be the velocity of projection; then with the figure of Art. 51 we have at the end of the time $t$

$$MP = AT = ut.$$

This shews that whatever be the direction of projection all the particles at the end of the time $t$ are on the surface of a sphere of which the radius is $ut$, and the centre is $M$; so that the centre is at the distance $\frac{1}{2}gt^2$ below the point of projection.

74. *If two particles are projected from the same point at the same instant, with different velocities, and in different directions, the straight line which joins them will always move parallel to itself.*




Let $A$ be the point of projection; suppose one body projected along $AP$, and the other along $AQ$.

First suppose the force of gravity not to exist. Then each body would *move uniformly* in a straight line. Suppose one body

to be at $P$ at the end of the time $T$, and at $p$ at the end of the time $t$; and suppose that the other body is at $Q$ at the end of the time $T$, and at $q$ at the end of the time $t$.

Then $$\frac{AP}{Ap}=\frac{T}{t}=\frac{AQ}{Aq};$$

therefore $PQ$ is parallel to $pq$, by Euclid, VI. 2. Now suppose the force of gravity to act; then in the time $t$ each body would be drawn down through the vertical space $\frac{1}{2}gt^2$. Thus take $pR$ and $qS$ vertically downwards, and each equal to $\frac{1}{2}gt^2$; then $R$ and $S$ are the positions of the bodies at the end of the time $t$; and $RS$ is parallel to $PQ$ by Euclid, XI. 9.

75. *If three particles are projected from the same point, at the same instant, with different velocities, and in different directions, the plane which passes through them always moves parallel to itself.*

This is demonstrated in the same manner as the preceding proposition; Euclid, XI. 15 will be required.

76. Let $v$ be the velocity at any point $P$ of the parabola described by a projectile; let $S$ be the focus: it is shewn in Art. 54 that $v^2=2gSP$. Let $p$ be the perpendicular from $S$ on the tangent to the parabola at $P$; then it is known from Conic Sections that $SP$ varies as $p^2$. Hence $v$ varies as $p$. And the perpendicular from $S$ on the tangent at $P$ is at right angles to the tangent, that is at right angles to the direction of the velocity at $P$.

Since then the perpendicular varies as the velocity and is always at right angles to the direction of the velocity, it may be conveniently used to furnish a representation of the velocity at any point of the parabola described by a projectile.

77. It may be shewn that the path of a projectile is a parabola in the following way:

Let $u$ be the velocity of projection, $a$ the angle which the direction of projection makes with the horizon. Then the vertical height of the body at the end of the time $t$ is $tu \sin a - \frac{1}{2}gt^2$; and therefore the distance of the body from a straight line parallel to the horizon and at the height $\frac{u^2}{2g}$ above the point of projection is $\frac{u^2}{2g} - tu \sin a + \frac{1}{2}gt^2$.

Now $$\left\{\frac{u^2}{2g} - tu \sin a + \frac{1}{2}gt^2\right\}^2$$

$$= \left\{\frac{u^2}{2g}(\sin^2 a - \cos^2 a) - tu \sin a + \frac{1}{2}gt^2 + \frac{u^2}{g}\cos^2 a\right\}^2$$

$$= \left\{\frac{u^2}{2g}(\sin^2 a - \cos^2 a) - tu \sin a + \frac{1}{2}gt^2\right\}^2$$

$$+ \frac{2u^2\cos^2 a}{g}\left\{\frac{u^2}{2g}(\sin^2 a - \cos^2 a) - tu \sin a + \frac{1}{2}gt^2\right\} + \left(\frac{u^2\cos^2 a}{g}\right)^2$$

$$= \left\{\frac{u^2}{2g}(\sin^2 a - \cos^2 a) - tu \sin a + \frac{1}{2}gt^2\right\}^2$$

$$+ t^2u^2\cos^2 a - \frac{2u^3t}{g}\cos^2 a \sin a + \frac{u^4}{g^2}\sin^2 a \cos^2 a$$

$$= \left\{\frac{u^2}{2g}(\sin^2 a - \cos^2 a) - tu \sin a + \frac{1}{2}gt^2\right\}^2$$

$$+ \left\{tu \cos a - \frac{u^2}{g}\sin a \cos a\right\}^2.$$

Now the horizontal distance which the body moves through in the time $t$ is $tu \cos a$; and so the expression just given is the square of the distance of the body at the time $t$ from a certain fixed point, namely the point which is

at the height $\frac{u^2}{2g}(\sin^2 a - \cos^2 a)$ above the horizontal plane through the point of projection and at the horizontal distance $\frac{u^2}{g}\sin a \cos a$ from this point.

Thus we see that the distance of the body from a certain fixed straight line is always equal to its distance from a certain fixed point. Therefore from the *definition of a parabola* the path of a projectile must be a parabola.

Hence it follows that we could thus by the aid of mechanical principles *demonstrate* that the property employed in Art. 51 belongs to a parabola, without assuming it from geometry.

## EXAMPLES. VI.

1. Find the velocity and the direction of projection in order that a projectile may pass horizontally through a given point.

2. Find the velocity with which a body must be projected in a given direction from the top of a tower so as to strike the ground at a given point.

3. A body is projected with a given velocity at an inclination $a$ to the horizon; a plane inclined at an angle $\beta$ to the horizon passes through the point of projection: find the condition in order that the body when it strikes the plane may be at the highest point of its path.

4. Two bodies are simultaneously projected in the same vertical plane with velocities $u$ and $v$ at inclinations $a$ and $\beta$ to the horizon. Shew that their directions are parallel after the time $\frac{uv \sin(a-\beta)}{g(v\cos\beta - u\cos a)}$.

5. Bodies are projected from the same point in the same vertical plane and in such a manner that the parabolas have a latus rectum of given length: shew that the locus of the vertices of these parabolas is a parabola with a latus rectum of the same length.

6. Bodies are projected from the same point in the same vertical plane so as to describe parabolas having a latus rectum of given length: shew that the locus of the foci is a parabola with a latus rectum of the same length, having its vertex downwards, and its focus at the point of projection.

7. Bodies are projected simultaneously from the same point, and strike the horizontal plane through that point simultaneously: shew that the latera recta of the paths vary as the squares of the horizontal ranges.

8. A body is projected at an inclination $\alpha$ to the horizon: determine when the motion is perpendicular to a plane which is inclined at an angle $\beta$ to the horizon.

9. A body is projected at an inclination $\alpha$ to the horizon: determine the condition in order that the body may strike at right angles the plane which passes through the point of projection and makes an angle $\beta$ with the horizon.

10. A body is projected with the velocity $u$ and strikes at right angles a plane which passes through the point of projection and is inclined at an angle $\beta$ to the horizon: shew that the height of the point struck above the horizontal plane through the point of projection is

$$\frac{2u^2}{g}\,\frac{\sin^2\beta}{1+3\sin^2\beta}.$$

11. Shew by mechanical considerations that any diameter of a parabola bisects the chords which are parallel to the tangent at the extremity of the diameter.

12. Shew that the time of describing any arc of a parabola by a projectile is equal to the time of moving uniformly over the chord with the velocity which the projectile has when it is moving parallel to the chord.

13. The time of describing any arc of a parabola by a projectile is equal to twice the time of falling vertically from rest from the curve to the middle point of the chord.

14. Two bodies are projected from two given points in the same vertical line in parallel directions and with equal velocities: shew that tangents drawn to the path of the lower will cut off from the path of the upper arcs described in equal times.

15. A smooth plane of length $l$ is inclined at an angle $a$ to the horizon; a body is projected up the plane with the velocity $u$, and after leaving the plane describes a parabola: shew that the greatest vertical height reached above the point of projection is $l \sin a \cos^2 a + \frac{u^2}{2g} \sin^2 a$.

16. A heavy body is projected from a given point in a given vertical plane with a given velocity so as to pass through another given point: shew that the locus of the second point in order that there may be only one parabolic path is a parabola having the given point as focus.

17. A ball is shot from a cannon with velocity $v$, at an inclination $a$ to the horizon; the cannon is moving horizontally with velocity $u$ in a direction inclined at an angle $\beta$ to the vertical plane which is parallel to the cannon: find the range of the ball on the horizontal plane.

18. A stone is thrown in such a manner that it would just hit a bird on the top of a tree, and afterwards reach a height double that of the tree; if at the moment of throwing the stone the bird flies away horizontally, shew that the stone will notwithstanding hit the bird if the horizontal velocity of the stone be to that of the bird as $1 + \sqrt{2}$ is to 2.

19. Find the time in which a projectile would reach a plane inclined to the horizon at an angle equal to the angle of projection, and bisecting the range on the horizontal plane.

20. A particle is projected from the top of a tower at an inclination $a$ to the horizon, with the velocity which would be acquired in falling down $n$ times the height of the tower. Obtain a quadratic equation for determining the range on the horizontal plane through the foot of the tower.

21. If $h$ be the height of the tower in the preceding Example, shew that the greatest possible range is $2h\sqrt{(n^2+n)}$, and that the tangent of the corresponding angle of projection is $\sqrt{\frac{n}{n+1}}$.

22. A particle being projected with velocity $u$, at an inclination $a$, just clears a cube of which the edge is $c$, which stands on the horizontal plane: find the relation between $u$, $a$, and $c$.

23. From the result of the preceding Example form a quadratic equation for finding $\tan a$; and thence shew that the least possible value of $u^2$ is $3cg$.

24. In the last Example shew that when $u$ is least $\tan a = \sqrt{5}$, and that the point of projection is at the distance $\frac{c}{2}(\sqrt{5}-1)$ from the cube.

25. If $t$ be the time in which a projectile describes an arc of a parabola, and $v$ the velocity which a particle would acquire in falling from the intersection of tangents at the extremities of the arc to the chord of the arc, shew that

$$gt = v\sqrt{2}.$$

26. Shew that the greatest range up an inclined plane of $30^0$ is two-thirds of the greatest range on a horizontal plane, the initial velocity being the same in the two cases.

27. A number of heavy particles are projected simultaneously from a point; if tangents be drawn to their paths from any point in the vertical straight line through the point of projection, prove that the points of contact will be simultaneous positions of the particles.

28. Two projectiles start from the same point at the same instant with any velocities: prove that they will both be moving in a common tangent to their paths at the same instant after an interval of time which is the Arithmetical mean between the times in the two paths from the point of projection to the point where the paths meet again.

## VII. *Mass.*

78. The word *matter* is in common use; and it is not easy to define it so as to give a notion of it to any person who does not already possess the notion. The following definitions have been proposed:

*Body or matter is any thing extended, and possessing the power of resisting the action of force.*

*Matter is the Substance, Material, or Stuff, of which all bodies are composed that are capable of having forces applied to them.*

79. The word *mass* is used as an abbreviation for *quantity of matter.*

80. We assume that at the same place on the Earth's surface, the masses of bodies are proportional to their weights. We will explain the grounds of this assumption.

If we take a cubic inch of lead, we find by experiment that it produces the same effect by its weight as another cubic inch of lead; and thus two cubic inches of lead produce by their weight twice the effect which one cubic inch of lead produces by its weight. Now it is a very natural supposition that so long as we keep to one kind of substance the mass is proportional to the volume; and therefore, so long as we keep to the same kind of substance the mass is proportional to the weight. We assume then that this will also be true when we compare bodies which are not of the same kind of substance.

81. Now suppose we have two bodies containing equal volumes of the same kind of substance. If a certain force acting for a certain time on one of these bodies generates a certain velocity, an equal force acting for an equal time on the other body will generate an equal velocity. Imagine that the two bodies are united into one body, and that the two forces are made to act on the united body: it is most natural to conclude that a velocity equal to the former will still be generated in an equal time. We are thus

led to suppose that when bodies of different masses, but composed of substance of the same kind, are similarly acted on by forces proportional to the masses, the velocities generated in equal times will be equal.

Thus, as long as we keep to the same kind of substance, we see that in order to generate a certain velocity in a certain time, the force must vary as the mass. We assume that this is also true for bodies which are not of the same kind of substance.

We have already seen that for the same body the force varies as the velocity generated in a given time; and we now see that for the same velocity the force varies as the mass. Hence, by Algebra, when both the velocity and the mass vary the force varies as their product; or in other words, *when a force acts on a body, the product of the mass moved into the velocity generated in a given time is proportional to the force.*

82. We see then that the velocity generated in a given time by a given force, varies *inversely* as the mass. This fact, that the greater the mass the less the effect which a given force produces, is sometimes expressed by saying that matter is *inert*, or that *inertia* is a property of matter. The words *inert* and *inertia* however are sometimes used in reference to the fact involved in the First Law of Motion, namely, that a body cannot change its own state of rest or motion.

83. The word *momentum* is used as an abbreviation of the *product of the mass moved into the velocity.*

84. We now repeat the Second Law of Motion. *Change of motion is proportional to the acting force, and takes place in the direction of the straight line in which the force acts.*

By *motion* here we are to understand motion as measured by *momentum;* so that we can now remove the restriction of having only *one* body and *one* force, which we have hitherto regarded, and may proceed to those more complex cases in which different bodies and different forces occur.

85. One case of the general principle of Art. 82 will be as follows; the weight of a body at a given place is proportional to the product of the mass moved into the velocity generated in a given time. Let the given time be one second, and the unit of length one foot; then the velocity generated is denoted by $g$. Let $M$ be the mass of a body, and $W$ its weight; then $W$ varies as $Mg$, so that by Algebra $W = CMg$, where $C$ is some constant.

It is convenient to have this constant equal to unity; this we can secure by making a suitable connexion between the units of mass and of weight which have not yet been fixed: then $W = Mg$.

Suppose, for example, we resolve to have one lb. as the unit of weight: required to determine the unit of mass. Let $M = 1$; then we obtain $W = g$, that is $32 \cdot 2$; so that the unit of mass is so much mass as weighs $32 \cdot 2$ lbs.

Again, suppose, for example, we resolve to have the mass of one cubic foot of water as the unit of mass, required to determine the unit of weight. Let $W = 1$; then we obtain $M = \frac{1}{32 \cdot 2}$: so that the unit of weight is such a weight that its mass is $\frac{1}{32 \cdot 2}$, that is, the mass of the unit of weight is $\frac{1}{32 \cdot 2}$ of the mass of a cubic foot of water. Now it is known by experiment that a cubic foot of water weighs 1000 ounces, so that the unit of weight is $\frac{1000}{32 \cdot 2}$ ounces.

86. We may illustrate the preceding remarks by discussing the motion of a body sliding on a *rough* Inclined Plane.

Suppose a Plane inclined at an angle $\alpha$ to the horizon; let a body be placed on the Plane. Let $M$ denote the mass of the body, and therefore $Mg$ its weight. The resolved force of gravity down the Plane is $Mg \sin \alpha$. The pressure on the Plane is $Mg \cos \alpha$. If $\mu$ denote the coefficient of friction, the friction will be $\mu Mg \cos \alpha$.

If the body is moving *down* the Plane, the friction acts *up* the Plane. Hence the resultant force down the Plane is $Mg(\sin\alpha - \mu\cos\alpha)$. Now when a body is acted on by its own weight, the velocity generated in a unit of time is $g$; that is, the force $Mg$ generates in a body of mass $M$ the velocity $g$ in a unit of time: therefore, by Art. 81, the force $Mg(\sin\alpha - \mu\cos\alpha)$ will generate the velocity $g(\sin\alpha - \mu\cos\alpha)$ in a unit of time.

Thus the motion of a body sliding down a rough Inclined Plane is similar to that of a body sliding down a smooth Inclined Plane, or to that of a body falling freely: the *acceleration* is $g(\sin\alpha - \mu\cos\alpha)$ for the rough Plane, $g\sin\alpha$ for the smooth Plane, and $g$ for the body falling freely.

In the same manner it may be shewn that if a body is sliding *up* a rough Inclined Plane the acceleration is $g(\sin\alpha + \mu\cos\alpha)$ *downwards.*

87. We have then the following important general result: *if a force* F *act on a body of mass* M *the acceleration* is $\frac{F}{M}$. This result follows from Art. 82 by making as in Art. 85 a suitable connexion between the unit of force and the unit of mass.

## Examples. VII.

1. A body weighing $n$ lbs. is moved by a constant force which generates in the body in one second a velocity of $a$ feet per second: find the weight which the force could support.

2. Find in what time a force which would support a weight of 4 lbs., would move a weight of 9 lbs. through 49 feet along a smooth horizontal plane: and find the velocity acquired.

3. Find how far a force which would support a weight of $n$ lbs., would move a weight of $m$ lbs. in $t$ seconds: and find the velocity acquired.

4. Find the number of inches through which a force of one ounce constantly exerted will move a mass weighing one lb. in half a second.

5. Two bodies urged from rest by the same uniform force describe the same space, the one in half the time the other does: compare their final velocities and their momenta.

6. If a weight of 8 lbs. be placed on a plane which is made to descend vertically with an acceleration of 12 feet per second, find the pressure on the plane.

7. If a weight of $n$ lbs. be placed on a plane which is made to ascend vertically with an acceleration $f$, find the pressure on the plane.

8. Find the unit of time when the unit of space is two feet, and the unit of weight is the weight of a unit of mass; assuming the equation $W=Mg$.

9. A body is projected up a rough Inclined Plane, with the velocity which would be acquired in falling freely through 12 feet, and just reaches the top of the Plane; the inclination of the Plane to the horizon is $60^0$, and the coefficient of friction is equal to $\tan 30^0$: find the height of the Plane.

10. A body is projected up a rough Inclined Plane with the velocity $2g$; the inclination of the Plane to the horizon is $30^0$, and the coefficient of friction is equal to $\tan 15^0$: find the distance along the Plane which the body will describe.

11. A body is projected up a rough Inclined Plane; the inclination of the Plane to the horizon is $\alpha$, and the coefficient of friction is $\tan \epsilon$: if $m$ be the time of ascending, and $n$ the time of descending, shew that $\left(\frac{m}{n}\right)^2 = \frac{\sin(\alpha - \epsilon)}{\sin(\alpha + \epsilon)}$.

12. Find the locus of points in a given vertical plane from which the times of descent down equally rough Inclined Planes to a fixed point in the vertical plane vary as the lengths of the Planes.

## VIII. *Third Law of Motion.*

88. Newton's Third Law of Motion is thus enunciated:

*To every action there is always an equal and contrary reaction: or the mutual actions of any two bodies are always equal and oppositely directed in the same straight line.*

Newton gives three illustrations of this Law:

If any one presses a stone with his finger, his finger is also pressed by the stone.

If a horse draws a stone fastened to a rope, the horse is drawn backwards, so to speak, equally towards the stone.

If one body impinges on another and changes the motion of the other body, its own motion experiences an equal change in the opposite direction. Motion here is to be understood in the sense explained in Art. 84.

The first of Newton's illustrations relates to forces in Statics; and the law of the equality of action and reaction in the sense of this illustration has been already assumed in this work; see *Statics*, Art. 286. The second illustration applies to a class of cases of motion which we shall consider in the present Chapter. The third illustration applies to what are called *impulsive forces*, which we shall consider in the next Chapter.

89. *Two heavy bodies are connected by a string which passes over a fixed smooth Pully: required to determine the motion.*

Let $m$ be the mass of the heavier body, and $m'$ the mass of the other. Let $T$ be the tension of the string, which is the same throughout by the Third Law of Motion, the weight of the string being neglected as usual.

The forces which act on each body are its weight and the tension of the string; and these forces act in opposite

directions. Thus the resultant force on the heavier body is $mg - T$ downwards, and on the lighter body $T - m'g$ upwards. Therefore the acceleration on the heavier body is $\frac{mg - T}{m}$, and on the lighter body $\frac{T - m'g}{m'}$. (Art. 87.)

Now as the string is supposed to be inextensible, the two bodies have at every instant equal velocities; and therefore the accelerations must be equal. Thus

$$\frac{mg - T}{m} = \frac{T - m'g}{m'};$$

therefore

$$T = \frac{2gmm'}{m + m'}.$$

Hence the acceleration is

$$g - \frac{2gm'}{m + m'}, \text{ that is } \frac{m - m'}{m + m'} g.$$

This is a constant quantity. Hence the motion of the descending body is like that of a body falling freely, but is not so rapid; for instead of $g$ we have now $\frac{m - m'}{m + m'} g$.

If $m = m'$ there is no acceleration; and so if there is any motion it is a uniform motion.

90. In the investigation of the preceding Article no notice is taken of the motion of the Pully: thus the result is not absolutely true. But it may be readily supposed that if the mass of the Pully be small compared with that of the two bodies, the error is very slight; and the supposition is shewn to be correct in the higher parts of Dynamics. Theoretically instead of a Pully, we might have a smooth peg for the string to pass round, but practically it is found that owing to friction this arrangement is not so suitable: see *Statics*, Arts. 191 and 281.

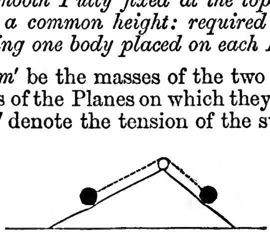
91. The system of two bodies considered in Art. 89 forms the essential part of a machine devised by Atwood, for testing experimentally the results obtained with respect

to rectilinear motion under the action of uniform forces. Atwood's machine contains some contrivances for diminishing friction, and some for assisting in the arrangement and observation of the experiments; but the principle is not affected by these contrivances.

The chief advantage secured by Atwood's machine is that by taking two bodies of nearly equal weight we can make $\frac{m-m'}{m+m'}g$ as small as we please, and thus render the motion slow enough to be observed without difficulty. The results of experiments with Atwood's machine are found to agree with those assigned by the investigations already given; and thus they confirm the two important statements that $g$ is constant at the same place, and that its value at London is about 32·2.

92. *Two bodies are connected by a string, which passes over a small smooth Pully fixed at the top of two Inclined Planes having a common height: required to determine the motion, supposing one body placed on each Plane.*

Let $m$ and $m'$ be the masses of the two bodies; $a$ and $a'$ the inclinations of the Planes on which they are respectively placed. Let $T$ denote the tension of the string.



Suppose the body of mass $m$ to be descending. The weight of this body is $mg$; the resolved part of the weight along the Plane is $mg \sin a$; hence the resultant force down the plane is $mg \sin a - T$, and therefore the acceleration is $\frac{mg \sin a - T}{m}$.

Similarly, for the other body, the resultant force up the Plane on which it moves is $T - m'g \sin a'$, and the acceleration is $\frac{T - m'g \sin a'}{m'}$.

Now as the string is supposed to be inextensible, the two bodies have at every instant equal velocities: and therefore the accelerations must be equal. Thus

$$\frac{mg \sin \alpha - T}{m} = \frac{T - m'g \sin \alpha'}{m'};$$

therefore
$$T = \frac{mm'g(\sin \alpha + \sin \alpha')}{m + m'}.$$

Hence the acceleration is

$$g \sin \alpha - \frac{m'g(\sin \alpha + \sin \alpha')}{m + m'}, \text{ that is } \frac{m \sin \alpha - m' \sin \alpha'}{m + m'} g.$$

Thus we see, that in order that this may be positive, and so the body of mass $m$ be acquiring downward velocity, we must have $m \sin \alpha$ greater than $m' \sin \alpha'$.

If $m \sin \alpha = m' \sin \alpha'$ there is no acceleration; and so if there is any motion it is a uniform motion.

## Examples. VIII.

1. If the two weights in Art. 89 are 15 ounces and 17 ounces respectively, find the space described and the velocity acquired in five seconds from rest.

2. If the string in Art. 89 were cut at the instant when the velocity of each body is $u$, find the distance between the two bodies after a time $t$.

3. In the system of Art. 89 shew that if the sum of the weights be given, the tension is greater the less the acceleration is.

4. A weight $P$ is drawn along a smooth horizontal table by a weight $Q$ which descends vertically, the weights being connected by a string passing over a smooth Pully at the edge of the table: determine the acceleration.

5. A weight $P$ is drawn up a smooth plane inclined at an angle of $30^0$ to the horizon, by means of a weight $Q$ which descends vertically, the weights being connected

by a string passing over a small Pully at the top of the plane: if the acceleration be one-fourth of that of a body falling freely, find the ratio of $Q$ to $P$.

6. Two weights $P$ and $Q$ are connected by a string; and $Q$ hanging over the top of a smooth plane inclined at $30^0$ to the horizon, can draw $P$ up the length of the plane in just half the time that $P$ would take to draw up $Q$: shew that $Q$ is half as heavy again as $P$.

7. Four equal weights are fastened to a string: find how they must be arranged so that when the string is laid over a fixed smooth Pully, the motion may be the same as that produced when two of the weights are drawn over a smooth horizontal table by the weight of the other two hanging over the edge of the table.

8. Two weights of 5 lbs. and 4 lbs. together pull one of 7 lbs. over a smooth fixed Pully, by means of a connecting string; and after descending through a given space the 4 lbs. weight is detached and taken away without interrupting the motion: find through what space the remaining 5 lbs. weight will descend.

9. Two weights are attached to the extremities of a string which is hung over a smooth Pully, and the weights are observed to move through 6·4 feet in one second; the motion is then stopped, and a weight of 5 lbs. is added to the smaller weight, which then descends through the same space as it ascended before in the same time: determine the original weights.

10. Find what weight must be added to the smaller weight in Art. 89, so that the acceleration of the system may have the same numerical value as before, but may be in the opposite direction.

11. Solve the problem in Art. 92, supposing the Inclined Planes rough.

12. If the Pully in Art. 89 can bear only half the sum of the weights of the two bodies, shew that the weight of the heavier body must not be less than $(3+2\sqrt{2})$ times the weight of the lighter body.

## IX. *The Direct Collision of Bodies.*

93. We have hitherto spoken of force as measured by the momentum which it generates in a given time; and the force with which we are most familiar is that of gravity, which takes an appreciable time to generate in any body a moderate velocity. There are however examples of forces which generate or destroy a large velocity in a time which is too brief to be appreciated. For example, when a cricket ball is driven back by a blow from a bat, the original velocity of the ball is destroyed, and a new velocity generated; and the whole time of the action of the bat on the ball is extremely brief. Similarly when a bullet is discharged from a gun, a large velocity is generated in an extremely brief time. Forces which produce such effects as these are called *impulsive forces*, and the following is the usual definition: *An impulsive force is a force which produces a finite change of motion in an indefinitely brief time.*

94. Thus an impulsive force does not differ in *kind* from other forces, but only in *degree:* an impulsive force is a force which acts with great intensity during a very brief time.

As the Laws of Motion may be taken to be true whatever may be the intensity of the forces which produce or change the motion, we can apply these laws to impulsive forces. But since the duration of the action of an impulsive force is too brief to be appreciated, we cannot measure the force by the momentum generated in any given time: it is usual to state that an impulsive force is measured by the whole momentum which it generates.

95. We shall not have to consider the simultaneous operation of ordinary forces and impulsive forces for the following reason: the impulsive forces are so much more intense than the ordinary forces, that during the brief time of simultaneous operation, an ordinary force does not

produce an effect comparable in amount with that produced by an impulsive force. Thus, to make a supposition which is not extravagant, an impulsive force might generate a velocity of 1000 in less time than one-tenth of a second, while gravity in one-tenth of a second would generate a velocity of about 3.

96. The student might perhaps anticipate that difficulties would arise in the discussion of questions relating to impulsive forces, but it will appear as we proceed that the cases which we have to consider are sufficiently simple.

We may observe that the words *impact* and *impulse* are often used as abbreviations for *impulsive action.*

97. We are about to solve some problems relating to the collision of two bodies; the bodies may be considered to be small spheres of uniform density, and, as before, we take no account of any possible *rotation:* see Art. 10. The collision of spheres is called *direct* when at the instant of contact the centres of the spheres are moving in the straight line in which the impulse takes place; the collision of spheres is called *oblique* when this condition is not fulfilled.

98. When one body impinges directly on another, the following is considered to be the nature of the mutual action. The whole duration of the impact is divided into two parts. During the first part a certain impulsive force acts in opposite directions on the two bodies, of such an amount as to render the velocities equal. During the second part another impulsive force acts on each body in the same direction respectively as before, and the magnitude of this second impulsive force bears to that of the former a ratio which is constant for any given pair of substances. This ratio lies between the limits zero and unity, both inclusive. When the ratio is unity the bodies are called *perfectly elastic;* when the ratio is greater than zero and less than unity the bodies are called *imperfectly elastic;* and when the ratio is zero the bodies are called *inelastic.* The ratio is called the *coefficient of elasticity*, or the *index of elasticity.*

**99.** There are three assumptions involved in the preceding Article.

We assume that there is an epoch at which the velocities of the two bodies are equal; this will probably be admitted as nearly self-evident.

We assume that during each of the two parts into which the whole duration of the impact is divided by this epoch, the action on one body is equal and opposite to the action on the other; this is justified by the Third Law of Motion.

We assume that the action on each body after the epoch is in the same direction as before, and bears a certain constant ratio to it; this assumption may be taken for the present as an hypothesis, which is to be established by comparing the results to which it leads with observation and experiment. See Art. 104.

100. We have still to explain why the words *elastic* and *inelastic* are used in Art. 98. It appears from experiment that bodies are compressible in various degrees, and recover more or less their original forms after the compression has been withdrawn: this property is termed *elasticity*. When one body impinges on another, we may naturally suppose that the surfaces near the point of contact are compressed during the first part of the impact, and that they recover more or less their original forms during the second part of the impact.

101. *A body impinges directly on another: required to determine the velocities after impact, the elasticity being imperfect.*

Let a body whose mass is $m$, moving with a velocity $u$, impinge directly on another body whose mass is $m'$, moving with a velocity $u'$. Let $R$ denote the impulsive force which during the first part of the impact acts on each body in opposite directions. Then at the end of the first part of the impact, the momentum of the body of mass $m$ is $mu - R$, and therefore its velocity is $\dfrac{mu - R}{m}$: and the

momentum of the body of mass $m'$ is $m'u'+R$, and therefore its velocity is $\frac{m'u'+R}{m'}$. These velocities are equal by hypothesis, that is

$$\frac{mu-R}{m}=\frac{m'u'+R}{m'},$$

therefore
$$R=\frac{mm'(u-u')}{m+m'}.$$

Let $e$ denote the *index of elasticity*; then during the second part of the impact an impulsive force $eR$ acts on each body in the same direction respectively as before. Let $v$ denote the final velocity of the body of mass $m$, and $v'$ that of the body of mass $m'$; then

$$v=\frac{mu-(1+e)R}{m}=u-\frac{(1+e)m'(u-u')}{m+m'}$$

$$=\frac{mu+m'u'-em'(u-u')}{m+m'},$$

$$v'=\frac{m'u'+(1+e)R}{m'}=u'+\frac{(1+e)m(u-u')}{m+m'}$$

$$=\frac{mu+m'u'+em(u-u')}{m+m'}.$$

102. From the general formulæ of the preceding Article many particular results may be deduced; we will give some examples.

If the bodies are *perfectly elastic*, $e=1$; then we have

$$v=\frac{(m-m')u+2m'u'}{m+m'},\qquad v'=\frac{2mu-(m-m')u'}{m+m'}.$$

If the bodies are *inelastic*, $e=0$; then we have

$$v=v'=\frac{mu+m'u'}{m+m'}.$$

Again, suppose $u'=0$, so that a body of mass $m$, moving with a velocity $u$, impinges on a body of mass $m'$ at rest; then we have

$$v=\frac{m-em'}{m+m'}u, \qquad v'=\frac{m(1+e)}{m+m'}u.$$

Thus the body which is struck goes onwards, and the striking body goes onwards, or stops, or goes backwards, according as $m$ is greater than, equal to, or less than $em'$. If $m'=em$, then $v=(1-e)u$, and $v'=u$.

103. The formulæ of Art. 101 supply two important inferences. Multiply the value of $v$ by $m$, and the value of $v'$ by $m'$, and add; thus we obtain

$$mv+m'v'=mu+m'u'.$$

This is usually expressed by saying that *the momentum of the system is the same after impact as before.* It will be seen that by the *momentum of the system*, we mean the result obtained by the algebraical addition of the momentum of each body.

Again, subtract the value of $v'$ from that of $v$; thus we obtain

$$v-v'=-e(u-u').$$

This is usually expressed by saying that *the relative velocity after impact is* $-e$ *times the relative velocity before impact.* It will be seen that by the *relative velocity*, we mean the algebraical excess of the velocity of the one body over that of the other.

104. The results of the preceding Article have been deduced from the principles assumed in Art. 98: if these results were contradicted by observation and experiment we should infer that the principles are partly or entirely inadmissible. On the other hand, assuming these results to be confirmed by observation and experiment, we may proceed to examine what support is thus furnished to the principles.

The first result in the preceding Article may be put in the form $m(u-v)=m'(v'-u')$. This furnishes a corroboration of the truth of the Third Law of Motion; for it

shews that the whole force which has acted on one body is equal and opposite to that on the other.

We now pass to the second result. Let $R$ denote, as before, the impulsive action between the two bodies during the first part of the impact, and $R'$ that during the second part of the impact: we shall shew that it *will follow that* R′ *bears a constant ratio to* R.

For since $v$ and $v'$ are the respective velocities at the end of the impact we have

$$v = \frac{mu - (R + R')}{m}, \qquad v' = \frac{m'u' + R + R'}{m'};$$

therefore $$v - v' = u - u' - \left(\frac{1}{m} + \frac{1}{m'}\right)(R + R').$$

Now let us suppose that experience shews that for the same pair of substances $v - v'$ is always equal to $-e(u - u')$.

Then $$(u - u')(1 + e) = \left(\frac{1}{m} + \frac{1}{m'}\right)(R + R').$$

But $R = \dfrac{mm'(u - u')}{m + m'}$, therefore $R' = \dfrac{emm'(u - u')}{m + m'} = eR.$

This is the required result.

105. *A body impinges directly on another: required to determine the conditions in order that the bodies should interchange velocities.*

Using the same notation as before, we require that $v = u'$, and $v' = u$. Hence, by Art 103,

$$mu' + m'u = mu + m'u',$$

and $$u' - u = -e(u - u').$$

The first of these conditions may be written in this form: $(m - m')(u - u') = 0$. Hence we must have $m = m'$. The second condition shews that we must have $e = 1$. Thus *the bodies must be of equal mass and perfectly elastic.*

106. In Art. 101 we supposed the collision to be caused by one body *overtaking* the other. If the bodies

move originally in *opposite* directions, the collision will be caused by one body *meeting* the other; the investigation in this case will be similar to that already given, and the results will coincide with those which would be obtained by changing the sign of $u'$ in the formulæ for $v$ and $v'$.

107. The product of the mass of a body into the square of its velocity is called the *vis viva* of the body. The vis viva of a system of bodies is the sum of the vis viva of every body of the system. In the higher parts of Dynamics the consideration of vis viva frequently occurs; and it is usual in the elementary parts to demonstrate one proposition respecting vis viva: this will form the next Article.

108. *By the direct collision of two imperfectly elastic bodies the vis viva of the system is diminished.*

Let $u$ and $u'$ be the velocities before impact of two bodies whose masses are $m$ and $m'$ respectively, and $v$ and $v'$ their velocities after impact. Then by Art. 103,

$$mv + m'v' = mu + m'u', \qquad v - v' = -e(u - u').$$

Therefore $\quad (mv + m'v')^2 = (mu + m'u')^2,$

$$mm'(v - v')^2 = mm'e^2(u - u')^2$$
$$= mm'(u - u')^2 - mm'(1 - e^2)(u - u')^2;$$

therefore, by addition, $\quad (m + m')(mv^2 + m'v'^2)$

$$= (m + m')(mu^2 + m'u'^2) - mm'(1 - e^2)(u - u')^2;$$

therefore $mv^2 + m'v'^2 = mu^2 + m'u'^2 - \dfrac{mm'}{m + m'}(1 - e^2)(u - u')^2.$

Now $e$ cannot be greater than unity, so that $1 - e^2$ cannot be negative; hence $mv^2 + m'v'^2$ is always less than $mu^2 + m'u'^2$ except when $e = 1$, and then the two expressions are equal. The expression for the vis viva after impact shews that during compression vis viva to the amount of $\dfrac{mm'}{m + m'}(u - u')^2$ is lost; and then during restitution $e^2$ times this amount is regained.

109. It is usual to give the following example of the subject of the present Chapter: Let $A$, $B$, $C$ denote the masses of three bodies, such that the first and third are formed of the same substance; let $e$ be the index of elasticity for the first and second bodies, and therefore also for the second and third. Suppose the first body to impinge directly with the velocity $u$ on the second at rest; then the second acquires the velocity $\frac{A(1+e)u}{A+B}$. Suppose that the second body now impinges directly with this velocity on the third at rest; then the third acquires the velocity $\frac{B(1+e)}{B+C} \cdot \frac{A(1+e)}{A+B} u$, that is $\frac{AB(1+e)^2 u}{(A+B)(B+C)}$. *Now supposing every quantity given except* B, *required to determine* B *so that the velocity communicated to* C *may be the greatest possible.*

We have to make $\frac{B}{(A+B)(B+C)}$ as great as possible.

$$\frac{B}{(A+B)(B+C)} = \frac{B}{B^2+(A+C)B+AC} = \frac{1}{B+A+C+\frac{AC}{B}}.$$

We must therefore make the denominator of the last fraction as small as possible.

But $B+A+C+\frac{AC}{B} = \left(\sqrt{B} - \sqrt{\frac{AC}{B}}\right)^2 + (\sqrt{A}+\sqrt{C})^2$,

so that the least value is that when $\sqrt{B} - \sqrt{\frac{AC}{B}}$ vanishes, that is when $B = \sqrt{(AC)}$.

Hence the velocity communicated to the third body is greatest when the mass of the second body is a mean proportional between the masses of the first and third.

110. The theory of the collision of bodies appears to be chiefly due to Newton, who made some experiments and recorded the results: see the *Scholium* to the Laws of Motion in Book I. of the Principia. In Newton's experiments however the two bodies seem always to have been formed of the same substance. He found that the value

of $e$ for balls of worsted was about $\frac{5}{9}$, for balls of steel about the same, for balls of cork a little less, for balls of ivory $\frac{8}{9}$, for balls of glass $\frac{15}{16}$.

An extensive series of experiments was made by Mr Hodgkinson, and the results are recorded in the *Report of the British Association* for 1834. These experiments shew that the theory may be received as satisfactory, with the exception that the value of $e$, instead of being quite constant, diminishes when the velocities are made very large.

## EXAMPLES. IX.

1. An inelastic body impinges on another of twice its mass at rest: shew that the impinging body loses two-thirds of its velocity by the impact.

2. A body weighing 5 lbs. moving with a velocity of 14 feet per second, impinges on a body weighing 3 lbs., and moving with a velocity of 8 feet per second: find the velocities after impact supposing $e=\frac{1}{3}$.

3. Two bodies are moving in the same direction with the velocities 7 and 5; and after impact their velocities are 5 and 6: find the index of elasticity, and the ratio of the masses.

4. Two bodies of unequal masses moving in opposite directions with momenta numerically equal meet: shew that the momenta are numerically equal after impact.

5. A body weighing two lbs. impinges on a body weighing one lb.; the index of elasticity is $\frac{1}{2}$: shew that $2v=u+u'$, and that $v'=u$.

6. The result of an impact between two bodies moving with numerically equal velocities in opposite directions is that one of them turns back with its original velocity, and the other follows it with half that velocity: shew that one body is four times as heavy as the other, and that $e=\frac{1}{4}$.

7. Find the necessary and sufficient condition in order that $v'$ may be equal to $u$.

8. $A$ strikes $B$, which is at rest, and after impact the velocities are numerically equal: if $r$ be the ratio of $B$'s mass to $A$'s mass, shew that the index of elasticity is $\frac{2}{r-1}$, and that $B$'s mass is at least three times $A$'s mass.

9. $A$, $B$ and $C$ are the masses of three bodies, which are formed of the same substance; the first impinges on the second at rest, and then the second impinges on the third at rest: determine the index of elasticity in order that the velocity communicated to $C$ may be the same as if $A$ impinged directly on $C$.

10. A body impinges on an equal body at rest: shew that the vis viva before impact cannot be greater than twice the vis viva of the system after impact.

11. A series of perfectly elastic bodies are arranged in the same straight line; one of them impinges on the next, then this on the next, and so on: shew that if their masses form a Geometrical Progression of which the common ratio is $r$, their velocities after impact form a Geometrical Progression of which the common ratio is $\frac{2}{r+1}$.

12. A number of bodies $A, B, C, \ldots$ formed of the same substance, are placed in a straight line at rest. $A$ is then projected with a given velocity so as to impinge on $B$; then $B$ impinges on $C$; and so on. Find the masses of the bodies $B, C, \ldots$ so that each of the bodies $A, B, C, \ldots$ may be at rest after impinging on the next; and find the velocity of the $n^{\text{th}}$ ball just after it has been struck by the $(n-1)^{\text{th}}$ ball.

## X. *The Oblique Collision of Bodies.*

111. In the present Chapter we shall consider the *oblique* collision of bodies; see Art. 97. It will be found that the problems discussed involve only a more extensive application of principles already explained. We shall confine ourselves to cases in which the line of impact and the directions of the motions of the bodies are *in one plane.*

112. *A body impinges obliquely on another: required to determine the velocities after impact, the elasticity being imperfect.*

Let a body whose mass is $m$, moving with a velocity $u$, impinge on another whose mass is $m'$, moving with a velocity $u'$. Let the direction of the first velocity make an angle $\alpha$ with the line of impact, and that of the second an angle $\alpha'$. After impact let the velocities be denoted by $v$ and $v'$, and the angles which their directions make with the line of impact by $\beta$ and $\beta'$.

Resolve all the velocities along the line of impact and at right angles to it. No impulsive force acts on the bodies in the direction at right angles to the line of impact, and so the velocities at right angles to the line of impact remain unchanged. Hence

$$v \sin \beta = u \sin \alpha \quad \text{.......................(1)},$$

$$v' \sin \beta' = u' \sin \alpha' \quad \text{.....................(2)}.$$

The velocities along the line of impact are affected just as they would be if the velocities in the other direction did not exist. Hence, proceding as in Art. 101, we obtain

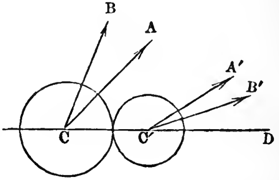
$$v \cos \beta = \frac{mu \cos \alpha + m'u' \cos \alpha' - em' (u \cos \alpha - u' \cos \alpha')}{m + m'} \text{....(3)},$$

$$v' \cos \beta' = \frac{mu \cos \alpha + m'u' \cos \alpha' + em (u \cos \alpha - u' \cos \alpha')}{m + m'} \text{....(4)}.$$

If we divide (1) by (3) we obtain the value of $\tan \beta$; this determines the *direction* of the velocity of the impinging

body after impact. If we square (1) and (3) and add, we obtain the value of $v^2$; this determines the *magnitude* of the velocity. Similarly from (2) and (4) we can determine the direction and the magnitude of the velocity of the other body after impact.

113. In the accompanying figure $C$ represents the centre of the body which we call the impinging body, considered to be a sphere, at the instant of impact; and $C'$ the centre of the other body. $CC'$ is the line of impact. The directions of the velocity of the impinging body before and after impact are represented by $CA$ and $CB$; and those of the other body by $C'A'$ and $C'B'$. Thus if $D$ be a point on $CC'$ produced,




$$\text{angle } ACD = \alpha, \quad \text{angle } A'C'D = \alpha',$$
$$\text{angle } BCD = \beta. \quad \text{angle } B'C'D = \beta'.$$

This figure may serve to illustrate the problem; it will however be easily perceived that the general formulæ admit of application to a large number of special cases, and that the figure would have to be modified in order to apply accurately to such special cases. For instance, we have supposed $CA$ and $C'A'$ to fall on the *same* side of $CD$, but it is of course possible that they should fall on *different* sides. It will be found on careful investigation that $u$ and $u'$ may always be considered to be *positive* quantities; and all the cases which can occur will be included in the general formulæ, where the angles have values lying between 0 and $180^0$, positive or negative.

The student should notice the particular results which may be deduced from the general formulæ as in Art 102.

114. Multiply equation (3) of Art. 112 by $m$, and equation (4) by $m'$ and add; thus we obtain

$$mv \cos \beta + m'v' \cos \beta' = mu \cos \alpha + m'u' \cos \alpha';$$

this shews that *the momentum of the system resolved in*

*the direction of the line of impact is the same after impact as before.*

The momentum of *each body* resolved in the direction *at right angles* to the line of impact is the same after impact as before, and therefore so also is the momentum of the system resolved in this direction.

Subtract equation (4) of Art. 112 from equation (3); thus we obtain

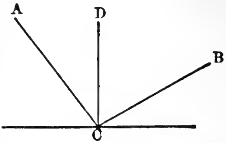
$$v \cos \beta - v' \cos \beta' = -e\,(u \cos \alpha - u' \cos \alpha').$$

This result may be expressed in words thus: *the relative velocity, resolved along the line of impact, after impact is* $-e$ *times its value before impact.*

115. We have hitherto treated of the collision of two bodies each of which is capable of motion; in the next Article we shall apply the principles already explained to a case of collision in which one body is fixed.

116. *A body impinges obliquely on a fixed smooth plane: required to determine the velocity after impact, the elasticity being imperfect.*

Let $m$ be the mass of the body. Let $AC$ represent the direction of the velocity before impact, meeting the plane at $C$, and $CB$ the direction after impact. Draw $CD$ at right angles to the plane; then, since the plane is smooth, $CD$ represents the line of impact.




Let $u$ denote the velocity before impact, and $v$ that after impact; let $\alpha$ denote the angle $ACD$, and $\beta$ the angle $BCD$.

Resolve the velocities along the line of impact and at right angles to it. No impulsive force acts on the body at right angles to the line of impact, and so the velocity at right angles to the line of impact remains unchanged. Hence

$$v \sin \beta = u \sin \alpha \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let $R$ denote the impulsive force which acts on the body during the first part of the impact. Then at the end of the first part of the impact the velocity of the body resolved along the line of impact is $\dfrac{mu\cos\alpha - R}{m}$; this is zero by hypothesis, therefore $R = mu\cos\alpha$. Let $e$ denote the index of elasticity; then during the second part of the impact an impulsive force $eR$ acts on the body; and therefore the final velocity along the line of impact

$$= \frac{mu\cos\alpha - (1+e)R}{m} = -\frac{eR}{m} = -eu\cos\alpha.$$

Thus

$$v\cos\beta = -eu\cos\alpha \text{.................}(2).$$

From (1) and (2) we obtain, by division,

$$\cot\beta = -e\cot\alpha \text{.................}(3).$$

The negative sign indicates that $CB$ and $CA$ are on opposite sides of $CD$, as represented in the figure: the velocity after impact along the line of impact, that is at right angles to the plane, is numerically $e$ times its value before impact. From (1) and (2) we have by squaring and adding

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha) \text{.................}(4).$$

Thus (3) determines the *direction* of the velocity after impact, and (4) determines its magnitude.

The angle $ACD$ is called the *angle of incidence*, and the angle $BCD$ the *angle of reflexion*. Thus from (3) we see that the cotangent of the angle of reflexion is always numerically equal to $e$ times the cotangent of the angle of incidence.

117. Some particular results of interest may be deduced from the preceding Article.

Suppose $e=1$; then $\cot\beta = -\cot\alpha$, and $v^2 = u^2$. Thus if the elasticity be perfect the angles of incidence and reflexion are numerically equal, and the velocities before and after impact are equal.

Suppose $e=0$; then $\beta$ is a right angle. Thus if there be no elasticity, the body after impact moves along the plane with the velocity $u \sin a$.

Suppose $a=0$, so that the impact is *direct*. Then after impact the body rebounds along its former course with $e$ times its former velocity.

Suppose $a=0$ and $e=0$. Then the body is brought to rest by the impact.

118. In the equations of Art. 112 suppose $u'=0$; then the equations become

$$v \sin \beta = u \sin a \quad \text{(1)},$$

$$v' \sin \beta' = 0 \quad \text{(2)},$$

$$v \cos \beta = \frac{m - em'}{m + m'} u \cos a \quad \text{(3)},$$

$$v' \cos \beta' = \frac{m(1+e)}{m+m'} u \cos a \quad \text{(4)}.$$

Let $\frac{m}{m'} = k$, so that $m = km'$; then the last two equations become

$$v \cos \beta = \frac{k - e}{1 + k} u \cos a, \qquad v' \cos \beta' = \frac{k(1+e)}{1+k} u \cos a.$$

Thus if $k$ be very small indeed we have very nearly

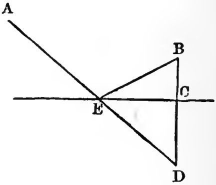
$$v \cos \beta = -eu \cos a \quad \text{(5)},$$

$$v' \cos \beta' = 0 \quad \text{(6)}.$$

Now the results (1) and (5) agree with those denoted by (1) and (2) in Art. 116. Thus we see that the case of a body impinging on a fixed plane is practically the same as that of a body impinging on another body of very much larger mass which is at rest. The comparison we have here made between the two cases is an example of a kind of exercise which is very valuable for students.

119. The theory of the collision of bodies gives the opportunity of forming a large number of illustrative problems; we will now solve some as examples.

120. *A body is to start from one given point, and after reflexion at a given fixed smooth plane it is to pass through another given point: required to determine the direction of incidence, the index of elasticity being supposed known.*




Let $A$ be the point from which the body is to start, $B$ the point through which the body is to pass after reflexion at the plane.

Draw $BC$ perpendicular to the plane, meeting it at $C$; produce $BC$ to $D$ so that $CD$ may be equal to $\frac{1}{e}BC$, where $e$ is the index of elasticity. Join $AD$, cutting the plane at $E$; then $AE$ is the required direction of incidence, and $EB$ is the direction of reflexion.

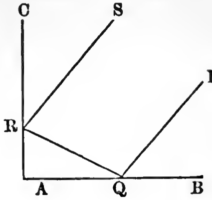
For the cotangent of the angle of incidence at $E$ is the tangent of $CED$, that is $\frac{CD}{CE}$; and the cotangent of the angle of reflexion at $E$ is the tangent of $BEC$, that is $\frac{BC}{CE}$. Therefore

$$\frac{\text{cotangent of the angle of reflexion}}{\text{cotangent of the angle of incidence}} = \frac{BC}{CD} = e.$$

Hence, by Art. 116, a body impinging on the plane in the direction $AE$ will be reflected in the direction $EB$.

121. *A body is reflected in succession by two fixed smooth planes, of the same substance, which are at right angles to each other, the body moving in a plane at right angles to the intersection of the fixed planes: required to shew that the directions of motion before the first reflexion and after the second reflexion are parallel.*

Let $PQRS$ be the course of the body, the first reflexion being at $Q$ and the second at $R$. Let $e$ be the index of elasticity.




Suppose the velocity before reflexion at $Q$ to consist of $u$ perpendicular to $AB$, and $v$ parallel to $AB$. After reflexion at $Q$ the velocity will consist of $-eu$ perpendicular to $AB$, and $v$ parallel to $AB$. After reflexion at $R$ the velocity will consist of $-eu$ perpendicular to $AB$, and $-ev$ parallel to $AB$.

Hence the value of each component velocity after reflexion at $R$ is $-e$ times its value before reflexion at $Q$. This shews that $RS$ is parallel to $QP$.

And the whole velocity after reflexion at $R$ is numerically equal to $e$ times the whole velocity before reflexion at $Q$.

We here assume that no force acts on the body during its motion except the impulses at $Q$ and $R$; so that we must suppose that gravity does not exist, or that it is practically neutralized by the motion taking place on a fixed smooth horizontal table.

122. If a body be projected in a direction inclined to the horizon it describes a parabolic arc; on reaching the ground it will in general rebound and describe another parabolic arc: we shall now investigate the connexion between these two arcs.

Let $u$ be the velocity of projection, and $a$ the inclination of the direction of projection to the horizon. Thus at starting the vertical velocity is $u \sin a$, and the horizontal velocity is $u \cos a$. The horizontal velocity is not changed during the motion. When the body reaches the ground its vertical velocity is the same as at starting; and accordingly it rebounds with a vertical velocity $eu \sin a$, if the ground be a smooth hard plane, and the index of elasticity be $e$.

Hence, on starting for the second parabolic arc, the body has the horizontal velocity $u \cos a$, and the vertical velocity $eu \sin a$; and all the circumstances of the motion can be determined; see Arts. 57, 58, 60, 62. Thus:

The latus rectum $= \dfrac{2u^2 \cos^2 a}{g}$, which is the same as that of the first arc.

The time of flight $= \dfrac{2eu \sin a}{g}$.

The greatest height reached $= \dfrac{e^2 u^2 \sin^2 a}{2g}$.

The range $= \dfrac{2}{g} eu \sin a u \cos a = \dfrac{2u^2}{g} e \sin a \cos a$.

After describing the second parabolic arc the body will rebound and describe a third parabolic arc; and so on. The following results are easily seen to hold:

All these parabolic arcs have the same latus rectum. The times of flight form a Geometrical Progression of which the common ratio is $e$. The greatest heights form a Geometrical Progression of which the common ratio is $e^2$. The ranges form a Geometrical Progression of which the common ratio is $e$.

123. In like manner if a projectile describe successive arcs by rebounding from an *inclined plane* which passes through the point of projection, it will be found that the times of flight form a Geometrical Progression of which the common ratio is $e$, and that the greatest distances from the inclined plane form a Geometrical Progression of which the common ratio is $e^2$.

124. *By the oblique collision of two imperfectly elastic bodies the vis viva of the system is diminished.*

Let $m$ and $m'$ be the masses of the bodies, $u$ and $u'$ their respective velocities before impact, $v$ and $v'$ their velocities after impact; let $a$ and $a'$ be the angles which

their directions of motion make with the line of impact before impact, $\beta$ and $\beta'$ the corresponding angles after impact.

Then, by Art. 114,

$$(mv\cos\beta + m'v'\cos\beta')^2 = (mu\cos a + m'u'\cos a')^2,$$

$$mm'(v\cos\beta - v'\cos\beta')^2 = mm'e^2(u\cos a - u'\cos a')^2$$

$$= mm'(u\cos a - u'\cos a')^2 - mm'(1-e^2)(u\cos a - u'\cos a')^2.$$

Hence by addition, and by division by $m+m'$,

$$mv^2\cos^2\beta + m'v'^2\cos^2\beta'$$

$$= mu^2\cos^2 a + m'u'^2\cos^2 a' - \frac{mm'(1-e^2)}{m+m'}(u\cos a - u'\cos a')^2.$$

Also $\qquad mv^2\sin^2\beta = mu^2\sin^2 a,$

and $\qquad m'v'^2\sin^2\beta' = m'u'^2\sin^2 a'.$

Therefore, by addition,

$$mv^2 + m'v'^2 = mu^2 + m'u'^2 - \frac{mm'(1-e^2)}{m+m'}(u\cos a - u'\cos a')^2;$$

and as $1-e^2$ cannot be negative the required result is obtained.

If the elasticity is perfect the vis viva of the system is the same after the collision as before.

## Examples. X.

[The elasticity is to be supposed imperfect unless the contrary is stated.]

1. A ball impinges on an equal ball at rest, the elasticity being perfect; if the original direction of the striking ball is inclined at an angle of $45^0$ to the straight line joining the centres, determine the angle between the directions of motion of the striking ball before and after impact.

2. A ball falls from a height $h$ on a horizontal plane, and then rebounds: find the height to which it rises in its ascent.

3. A ball falls from a height $h$ on a horizontal plane, and then rebounds, falls and rebounds again; and so on; find the sum of the spaces described.

4. A ball of mass $m$ impinges on a ball of mass $m'$ at rest: shew that the tangent of the angle between the old and new directions of motion of the impinging body is $\frac{1+e}{2} \frac{m' \sin 2a}{m + m'(\sin^2 a - e \cos^2 a)}$.

5. A ball of mass $m$ impinges on a ball of mass $m'$ at rest: find the condition which must hold in order that the directions of motion of the impinging ball before and after impact may be at right angles.

6. A ball impinges on an equal ball at rest, the elasticity being perfect; the angle between the old and new directions of motion of the impinging body is $60^0$: find the velocity after impact.

7. A ball impinges on an equal ball at rest, the elasticity being perfect: find the condition under which the velocities will be equal after impact.

8. A body is projected at an inclination $a$ to the horizon; and by continually rebounding from the horizontal plane describes a series of parabolas: find the tangent of the angle of projection at the $n^{\text{th}}$ rebound.

9. A body is projected with the velocity $u$, at the inclination $a$ to the horizon, and by continually rebounding from the horizontal plane describes a series of parabolas: find the sum of the ranges.

10. In the preceding Example find the time which elapses before the body ceases to rebound.

11. A ball is projected from a point in a smooth horizontal billiard table, and after striking the four sides in order returns to the starting point: shew that the sides of the parallelogram described are parallel to the diagonals of the table, the elasticity being perfect.

12. A ball is projected from the middle point of one side of a billiard table, so as to strike first an adjacent side, and then the middle point of the side opposite to that from which it started: determine the direction of projection.

13. Two balls moving in parallel directions with equal momenta impinge: shew that if their directions of motion be opposite they will move after impact in parallel directions with equal momenta.

14. In the preceding Example find the condition in order that the direction after impact may be at right angles to the original direction.

15. $A$ and $B$ are given positions on a smooth horizontal table: and $AC$, $BD$ are perpendiculars on a hard plane at right angles to the table. If a ball struck from $A$ rebounds to $B$ after an impact at the middle point of $CD$, shew that when the ball is sent back from $B$ to $A$, the point of impact on $CD$ will divide it into parts whose ratio is that of $e^2$ to 1.

16. $ABCD$ is an ordinary rectangular billiard table perfectly smooth; $E$ is a ball in a given position: it is required to select the proper position for another ball $F$ in all respects like the first, so that the player, striking $E$ on $F$, may cause $F$ to run into the corner pocket $A$, and $E$ to run into $D$ with equal velocities, the elasticity being perfect.

It is assumed that the *radius* of each ball may be neglected in comparison with the dimensions of the billiard table.

17. A ball is projected from a point between two vertical planes, the plane of motion being perpendicular to both: shew that the latera recta of the parabolic arcs described form a Geometrical Progression having the common ratio $e^2$.

18. A body slides down a smooth Inclined Plane of given height; at the bottom of the Inclined Plane the particle rebounds from a hard horizontal plane: find the range on the latter plane.

19. A ball is projected from a given point at a given inclination towards a vertical wall: determine the velocity of projection so that after striking the wall the ball may return to the point of projection.

20. Two equal balls start at the same instant with equal velocities along the diagonals of a square from the ends of a side, and when they meet, the line of impact is parallel to that side: determine the angle which the direction of motion of each ball after impact makes with the line of impact.

21. A perfectly elastic ball is projected with a given velocity from a point between two parallel walls, and returns to the point of projection after being once reflected at each wall: find the angle of projection.

22. An imperfectly elastic ball is thrown from a given point against a vertical wall: find the direction in which it must be projected with the least velocity, so as to return to the point of projection.

23. There are two parallel walls whose distance apart is equal to their height, and from the top of one wall a perfectly elastic ball is thrown horizontally so as to fall at the foot of the same wall after rebounding from the other: determine the position of the focus of the first path.

24. Bodies of different elasticities slide down a smooth Inclined Plane through the same vertical height, and impinge on a horizontal plane at its foot: shew that all the parabolas which are afterwards described have the same latus rectum.

25. A ball is projected in a given direction within a fixed horizontal hoop, so as to go on rebounding from the surface of the hoop: if the velocity at the end of every impact be resolved along the tangent and the normal to the hoop at the point, shew that the former component is constant, and that the latter component diminishes in Geometrical Progression.

26. Shew how to determine the direction of projection of a ball lying at a given point on a smooth billiard table, so that after striking all the sides in succession the ball may hit a given point.

## XI. *Motion of the Centre of Gravity of two or more bodies.*

125. We have explained in the Statics what is meant by the centre of gravity of a body or a system of bodies; and have shewn that for a given body or system there is only *one* centre of gravity. If a change takes place in the position of any body of the system, there is a corresponding change in the position of the centre of gravity of the system; and thus we are led to consider the motion of the centre of gravity of two or more bodies.

126. *Having given the velocities of two bodies estimated in any direction, required the velocity of their centre of gravity estimated in the same direction.*

Suppose $m$ and $m'$ the masses of the bodies; let their distances from a fixed plane at a certain instant be $a$ and $a'$ respectively; then the distance of the centre of gravity from the fixed plane is $\dfrac{ma + m'a'}{m + m'}$; see *Statics*, Arts. 119 and 146.

Let the velocities of the two bodies estimated at right angles to the plane be $b$ and $b'$; then at the end of a time $t$ the distances of the bodies from the fixed plane are $a + bt$ and $a' + b't$ respectively. Therefore the distance of the centre of gravity from the fixed plane

$$= \frac{m(a + bt) + m'(a' + b't)}{m + m'} = \frac{ma + m'a'}{m + m'} + \frac{mb + m'b'}{m + m'} t.$$

This shews that the distance of the centre of gravity from the fixed plane increases uniformly with the time; and that the velocity of the centre of gravity at right angles to the fixed plane is $\dfrac{mb + m'b'}{m + m'}$.

127. In the preceding Article we have assumed that the two bodies have *uniform* velocities in the assigned

direction; but the result may be easily extended to the case in which the velocities are not uniform. For the time $t$ may be as short as we please; and if the velocities of the bodies are really variable in the assigned direction, no error will ultimately arise from regarding them as uniform for an indefinitely short time. Thus we have the following general result: *the velocity of the centre of gravity of two bodies estimated in any direction at any instant is found by dividing the momentum of the system estimated in that direction at that instant by the sum of the masses.*

128. The result just enunciated for the case of *two* bodies is true for any number of bodies; the mode of demonstration is the same as that given for two bodies.

129. *The motion of the centre of gravity of two bodies is not affected by the collision of the bodies.*

First suppose the collision to be *direct.*

Let $m$ and $m'$ be the masses of the bodies, $u$ and $u'$ their velocities before impact, $v$ and $v'$ their velocities after impact. The velocity of the centre of gravity, by Art. 126, is $\frac{mu+m'u'}{m+m'}$ before impact, and $\frac{mv+m'v'}{m+m'}$ after impact; and these are equal by Art. 103.

Next suppose the collision to be *oblique.*

Let $m$ and $m'$ be the masses of the bodies, $u$ and $u'$ their velocities before impact, $\alpha$ and $\alpha'$ the angles which the directions of motion make with the line of impact; let $v$ and $v'$ be the corresponding velocities, and $\beta$ and $\beta'$ the corresponding angles after impact.

The velocity of the centre of gravity, estimated in the direction of the line of impact, by Art. 126, is

$$\frac{mu\cos\alpha+m'u'\cos\alpha'}{m+m'}$$

before impact, and is

$$\frac{mv\cos\beta+m'v'\cos\beta'}{m+m'}$$

after impact; and these are equal by Art. 114.

The velocity of the centre of gravity estimated in the direction at right angles to the line of impact, by Art. 126, is

$$\frac{mu \sin \alpha + m'u' \sin \alpha'}{m + m'}$$

before impact, and is

$$\frac{mv \sin \beta + m'v' \sin \beta'}{m + m'}$$

after impact; and these are equal by Art. 114.

Thus the component velocity of the centre of gravity in two directions is the same after impact as before; and therefore the resultant velocity is the same in magnitude and direction after impact as before.

130. It follows from the investigation of Art. 126, that if two bodies move in straight lines, each with uniform velocity, then their centre of gravity moves also in some straight line, with uniform velocity. Hence we may establish the following proposition: *the centre of gravity of two projectiles, which are moving simultaneously, describes a parabola.* For suppose at any instant that gravity ceased to act; then each body would move in a straight line with uniform velocity, and so would also the centre of gravity. The effect of gravity in a given time is to draw each body down a vertical space which is the *same* for each body, and which *varies as the square of the time:* and the centre of gravity is drawn down through the same vertical space. Hence, by reasoning as in Art. 51, we find that the path of the centre of gravity is a parabola.

And this result may be extended to the case of any number of bodies: see Art. 128.

131. By the method of Arts. 126 and 127, we may establish the following result: *If* f *and* f′ *be the accelerations, estimated in any direction, of two moving bodies, whose masses are* m *and* m′ *respectively, the acceleration of the centre of gravity of the two bodies estimated in the same direction is* $\frac{\mathrm{mf} + \mathrm{m'f'}}{\mathrm{m} + \mathrm{m'}}$.

And this result may be extended to the case of any number of bodies: see Art. 128.

## EXAMPLES. XI.

1. A body weighing 4 lbs. and another weighing 8 lbs. are moving in the same direction, the former with the velocity of 8 feet per second, and the latter with the velocity of 2 feet per second: determine the velocity of the centre of gravity.

2. Equal bodies start from the same point in directions at right angles to each other, one with the velocity of 8 feet per second, and the other with the velocity of 6 feet per second: determine the velocity of the centre of gravity.

3. In the system of Art. 89 supposing the initial velocity zero, find the velocity of the centre of gravity at the end of a given time.

4. A heavy body hanging vertically draws another along a smooth horizontal plane; supposing the initial velocity zero, find the horizontal and the vertical velocity of the centre of gravity at any instant.

5. Shew that the centre of gravity in the preceding Example describes a straight line with uniform acceleration.

6. In the system of Art. 92 supposing the initial velocity zero, find the velocity of the centre of gravity at the end of a given time resolved parallel to the two planes.

7. Shew that the centre of gravity in the preceding Example describes a straight line with uniform acceleration.

8. Two balls are dropped from two points not in the same vertical line, and strike against a horizontal plane, the elasticity being perfect: shew that the centre of gravity of the balls will never re-ascend to its original height, unless the initial heights of the balls are in the ratio of two square numbers.

9. Three equal particles are projected, each from one angular point of a triangle along the sides taken in order, with velocities proportional to the sides along which they move: shew that the velocity of the centre of gravity estimated parallel to each side is zero; and hence that the centre of gravity remains at rest.

10. $P$, $Q$, $R$ are points in the sides $BC$, $CA$, $AB$ respectively of the triangle $ABC$, such that $\frac{BP}{CP}=\frac{CQ}{AQ}=\frac{AR}{BR}$: shew that the centre of gravity of the triangle $PQR$ coincides with that of the triangle $ABC$.

## XII. *Laws of Motion. General Remarks.*

132. We propose in the present Chapter to make some general remarks concerning the Laws of Motion. It is not necessary that a student should devote much attention to this Chapter on his first reading of the subject. He should notice the points which are here considered, and when in his subsequent course he finds any difficulty as to these points he can examine the remarks which bear upon the difficulty.

133. We will here repeat the Laws of Motion.

I. Every body continues in a state of rest or of uniform motion in a straight line, except in so far as it may be compelled to change that state by force acting on it.

II. Change of motion is proportional to the acting force, and takes place in the direction of the straight line in which the force acts.

III. To every action there is always an equal and contrary reaction: or the mutual actions of any two bodies are always equal and oppositely directed in the same straight line.

It is manifest that instead of Laws of Motion it would be more accurate to call these statements, *Laws relating to the connexion of force with motion.*

134. We have already observed that the motion of a body here considered is of that kind in which all the points of the body describe curves identical in form, though varying in position. For example, when we speak of the motion of a falling body we mean such a motion that every point of the body describes a straight line. The motion which is here considered is called *motion of translation*, to distinguish it from *motion of rotation*, which we do not consider.

135. We have also stated, in connexion with the distinction just explained, that the Laws of Motion ought to be enunciated with reference to *particles* rather than to *bodies.* It might appear to a beginner that there can be

little advantage in studying the theory of the motion of particles, because in practice we are always concerned with bodies of finite size. But it is not difficult to shew the importance and value of a sound theory of the motion of particles. For it is easy to conceive that a solid body is made up of particles, and that the forces acting may be such as to render the motion of one particle exactly the same as the motion of another; and so the motion of the body is known when that of one particle is known. The case of a falling body illustrates this remark; see also Art. 81. Again, it is shewn in the higher parts of Mechanics that the motion of the centre of gravity of a rigid body is exactly the same as the motion of a particle having a mass equal to the mass of the rigid body, and acted on by forces equal and parallel to those which act on the rigid body. Although the student could not at the present stage follow the reasoning by which this remarkable result is obtained, nor even fully apprehend the result itself, yet he may readily perceive that great interest is thus attached to the theory of the motion of particles.

136. The terms *relative velocity* and *relative motion* are sometimes used in Mechanics; their meaning will be obvious from an illustration. Suppose two bodies, $A$ and $B$ moving along the same straight line, $A$ with the velocity of 4 feet per second, and $B$ foremost with the velocity of 6 feet per second. Then the distance between $A$ and $B$ increases at the rate of 2 feet per second; this rate is called the *relative velocity*. We may say that the motion of one body with respect to the other, or the *relative motion*, is the same as if $A$ were brought to rest, and $B$ moved *forwards* with the velocity of 2 feet per second; or, it is the same as if $B$ were brought to rest, and $A$ moved *backwards* with the velocity of 2 feet per second. See also Art. 103.

137. Up to the end of the sixth Chapter we considered the effect which a force produces on the velocity of a body without regard to the mass of the body moved. It is usual to apply the name *accelerating* force to force so considered; and hence the two following definitions are used:

Force considered only with respect to the velocity generated is called *accelerating force.*

Force considered with respect to the mass to which velocity is communicated as well as to the velocity generated is called *moving force*.

The terms tend to confuse a beginner, because they lead him to suppose that there are two kinds of force. There is really only one kind of force, namely that which is called *moving force* in the foregoing definitions; for when force acts it always acts on some body. It is not necessary to make any use of the term *accelerating force:* when the beginner hears or reads of an accelerating force $f$ he must remember that this means a force which produces the acceleration $f$ in the motion of the body which is considered.

138. We have followed Newton in our enunciation of the Laws of Motion; but it is necessary to observe that this course is not universally adopted. Many writers in effect divide Newton's Second Law into two, which they term the Second and Third Laws, presenting them thus:

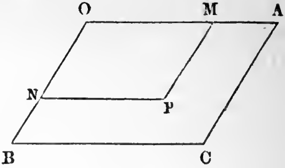
Second Law. When forces act on a body in motion each force communicates the same velocity to the body as if it acted singly on the body at rest.

Third Law. When force acts on a body the momentum generated in a unit of time is proportional to the force.

Then Newton's Third Law is presented as another principle which must be admitted to be true, although apparently not difficult enough or not important enough to be ranked formally with the Laws of Motion.

We have followed Newton for two reasons. In the first place, his mode of stating the Laws of Motion seems, to say the least, as good as any other which has been proposed: and in the second place, there is very great advantage in a uniformity among teachers and students as to the first principles of the subject, and this uniformity is more likely to be secured under the authority of Newton than under that of inferior names.

139. We have given in Art. 47 Newton's form of the *parallelogram of velocities;* some writers omit this, and supply its place by a *purely geometrical proposition*, which is substantially as follows:




Let $OACB$ be a parallelogram: from a point $P$ within it draw $PM$ parallel to $OB$, meeting $OA$ at $M$, and $PN$ parallel to $OA$, meeting $OB$ at $N$: then if a point moves in such a manner that $\frac{PM}{PN}=\frac{OB}{OA}$ always, $P$ must move along the diagonal $OC$.

Since $\frac{PM}{PN}=\frac{OB}{OA}$, it follows by Euclid, VI. 26, that the parallelograms $OMPN$ and $OACB$ are about the same diagonal. Thus $P$ must be on the straight line $OC$.

Also $P$ arrives at $C$ when $M$ is at $A$ and $N$ is at $B$; and if $M$ and $N$ move uniformly along $OA$ and $OB$ respectively, then $P$ moves uniformly along $OC$.

Thus we have demonstrated the result, without any reference to the notion of force, chiefly by the aid of Euclid, VI. 26. Students are sometimes perplexed by finding that while the theorem is asserted to be purely geometrical the enunciation and demonstration are expressed by the aid of language borrowed from Mechanics. In Newton's mode we arrive at the result as a deduction from the First and Second Laws of Motion. We have already seen, by an example given in Art. 77, that it is possible to obtain geometrical truths indirectly by the aid of Mechanics; and such a process is both interesting and valuable; but when we wish to draw special attention to the fact that a certain result is purely geometrical, it is advisable to restrict ourselves to geometrical language in the enunciation and investigation.

140. We have already stated that the direct experimental evidence for the truth of the Laws of Motion is not very strong; strictly speaking we might assert that there is no direct experimental evidence: for the Laws of Motion ought to be enunciated with respect to *particles*, and we

cannot make the requisite experiments on particles. In fact the Laws of Motion should be assumed in the outset as hypotheses, and their truth verified by the agreement of results deduced from them with accurate observations. We are enabled to institute some such comparisons by the aid of Atwood's machine; but, as we have said, it is from the close agreement of theory with observation in Astronomy that we derive the most convincing evidence of the truth of the Laws of Motion. The history of the progress of Mechanics confirms the statement that the Laws of Motion cannot be regarded as obviously true or even as readily admissible when enunciated. The Greeks excelled in Geometry, and were not ignorant of Statics; but even men so illustrious as Aristotle and Archimedes completely failed in their attempts at Dynamics; and the honour of laying the foundations of this subject was reserved for Galileo.

141. In Art. 51 we have devoted a few lines to shewing that $P$ is the position of the body at the end of the time $t$. It is usually considered sufficient to make the following statement: $AT$ is the space which a body moving with the velocity $u$ would describe in the time $t$, and $TP$ is the space through which the body would be drawn by gravity in the time $t$; and therefore by the Second Law of Motion $P$ is the position of the body at the end of the time $t$. This statement implies that the result is an immediate deduction from the Second Law of Motion. But the Second Law of Motion does not give us any immediate information about the *position* of a body when forces act on it; the Law is directly concerned only with the *velocity* of the body, and when we have determined the velocity of a body at any instant we have a further investigation to make in order to find the position of the body at any instant.

The point is perhaps of small importance in this case; but a beginner might easily be led into error on other occasions, if his attention had never been drawn to the exact meaning of the Second Law of Motion.

142. It will be interesting to give a brief account of that part of Newton's *Principia* which is devoted to the

Laws of Motion; the student will thus have his attention drawn to some important principles, which will be of service to him as he proceeds, although he may be unable at present to master them completely.

After enunciating and briefly illustrating the Laws of Motion, Newton adds a series of Corollaries; these we shall now state, omitting the commentary by which he supports them.

I. The proposition which is now called the Parallelogram of Velocities: see Art. 47.

II. The statical proposition which is now called the Parallelogram of Forces. Newton deduces this from his first Corollary, and points out some applications to the theory of machines.

III. The momentum of a system estimated in any direction is unaffected by the mutual actions of the bodies which compose the system. Newton considers principally the case of the collision of two bodies: see Art. 114.

IV. The position of the centre of gravity of two or more bodies is not changed by the mutual actions of the bodies; so that the centre of gravity of bodies acting on each other, and subject to no external forces, either remains at rest or moves uniformly in a straight line: see Art. 129.

V. The relative motions of bodies comprised within a given space are the same whether that space is at rest, or moving uniformly in a straight line. This is illustrated by the fact that motions take place in a ship in the same way whether the ship is at rest or moving uniformly in a straight line.

VI. If bodies be in motion in any manner their relative motions will not be changed if they are all acted on by forces producing equal accelerations in parallel directions. Arts. 74 and 75 illustrate this statement.

After the Corollaries Newton gives in a Scholium an account of experiments on the collision of bodies, and some additional remarks on the Third Law of Motion.

143. We have already stated that in our investigations respecting falling bodies, we leave out of consideration the resistance of the air; and that in consequence our results may in some cases deviate considerably from practical exactness. We will make some remarks on the nature of the resisting force exerted by the air.

Let us take for example the case of a falling body. It appears from experiments that the resistance of the air varies as the square of the velocity of the body; or at least this is very approximately the case. Let $v$ denote the velocity of the body; then the resistance of the air may be denoted by $kv^2$, where $k$ is some constant. Let $m$ be the mass of the body, and $mg$ the weight of the body; then the downward force on the body is $mg - kv^2$, and so the acceleration, at the instant the velocity is $v$, is $\frac{mg - kv^2}{m}$. Thus we see that the acceleration is not constant. In order to determine the motion of the body under this acceleration, more mathematical knowledge would be required than the student is at present supposed to possess; but two interesting results will be readily understood.

If there are two bodies of the same external form and substance, experiment shews that the coefficient $k$ is the same for both. Now the acceleration is $g - \frac{kv^2}{m}$; and therefore for a given value of $k$, the effect of the resistance is smaller the larger $m$ is. For example, suppose we have a solid sphere and a hollow sphere, made of the same substance, and having the same external radius; then the resistance of the air has less influence on the motion of the solid sphere than on the motion of the hollow sphere. Thus we are able to understand why the resistance of the air produces less effect on the motion of dense bodies, than on the motion of light bodies, other circumstances being the same.

If $kv^2 = mg$ the acceleration is zero, and the nearer the value of $v$ is to $\sqrt{\frac{mg}{k}}$ the smaller the acceleration becomes. This expression is called the *terminal velocity.*

If the body falls from rest its velocity will never exceed this value, but will approach indefinitely near to this value if the motion can continue long enough. If the body be projected downwards with a velocity greater than this expression the velocity will always exceed this value, but will approach indefinitely near to this value if the motion can continue long enough. Thus in each case the motion tends to become uniform.

144. The following example will illustrate the effect of the resistance of the air on falling bodies. In the fortress of Königstein in Saxony water is raised from a great depth below the surface of the ground. For the amusement of visitors a man draws up a bucket of water, and then pours the water back into the well. The depth is known to be about 640 feet, so that if there were no resistance from the air the sound of the splash should reach the ear in about 7 seconds; practically the time is about 15 seconds.

145. We have seen in Art. 43, that, if we neglect the resistance of the air, a body projected vertically upwards will take the same time in its descent as in its ascent, and will reach the ground with a velocity numerically the same as that at starting. These results will not hold when we take into account the resistance of the air; the time of ascent is then *less* than the time of descent, and the velocity on reaching the ground is *less* than the velocity at starting. The demonstration of these results will furnish a valuable exercise, and we will therefore give it.

The velocity on reaching the ground must be less than that at starting for two reasons: In the first place, in consequence of the resistance of the air, the body will not rise to so great a height as if there were no resistance; and therefore it falls down through a space *less* than that in which gravity, if unopposed, would generate a velocity equal to that in starting: and so if there were no resistance during the motion downwards the velocity on reaching the ground would be less than at starting. In the next place, while the body falls the resistance of the air *opposes* the action of gravity, and thus the velocity generated while the

body falls through any small space is less than that which would have been produced by the action of gravity alone, while the body falls through the same space: see equation (3) of Art. 40. Thus for both reasons the velocity on reaching the ground is less than the velocity at starting.

Again, in the same way it follows that the velocity at any point in the descent is *less* than it was at the *same point* in the ascent: and thus each indefinitely small part of the straight line described is moved over in less time in the ascent than in the descent; and therefore the whole time of ascent is less than the whole time of descent.

## EXAMPLES. XII.

1. If the velocities and directions of motion of two bodies moving in the same plane be known, find the direction and the magnitude of the velocity of one body relative to the other.

2. If $a$ be the distance at a given instant between two bodies which are moving uniformly, $V$ their relative velocity and $u$, $v$ the resolved parts of $V$ in, and at right angles to, the direction of $a$ respectively, shew that the distance of the bodies when they are nearest to each other is $\frac{av}{V}$, and find the time of arriving at this nearest distance.

3. Two bodies move with constant accelerations $f$ and $f'$ in given straight lines; they start with velocities $u$ and $u'$: find the relative velocity at the end of the time $t$ estimated along the straight line which makes angles $a$ and $a'$ with the directions of motion.

4. A ball is thrown up vertically with a velocity $u$ and meets with a uniform resistance equal to half the force of gravity both in the ascent and descent: if it reach the ground again with the velocity $v$, shew that $u=v\sqrt{3}$.

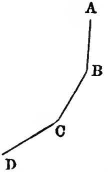
5. Two straight lines of railway cross each other at a given angle, and a train on each of them is approaching the junction, each with a given velocity: find geometrically or otherwise the velocity with which the trains are approaching each other.

## XIII. *Motion down a Smooth Curve.*

146. We shall now proceed to consider cases of motion in which the force acting is not constant in magnitude and direction; in the present Chapter we shall suppose a body to be acted on by two forces, namely, gravity and the resistance of a smooth fixed curve. The student may imagine a fine tube in the form of a curve, and a body in the shape of an indefinitely small sphere moving in the tube. We shall not attempt to determine the motion completely, for the mathematical difficulties would be too great for the student at present, but we shall demonstrate some important results.

147. *When a body descends down a smooth curve in a vertical plane the velocity acquired at any point is the same as if the body had fallen freely down the same vertical height.*

We shall consider the motion down a curve as the limiting case of the motion down an indefinitely great number of successive inclined planes.




Let $AB$, $BC$, $CD$,... represent successive inclined planes. Let $h_1$ be the vertical height of $A$ above $B$, $h_2$ the vertical height of $B$ above $C$, $h_3$ the vertical height of $C$ above $D$, and so on.

Suppose a body to slide down this series of planes. Let $v_1$, $v_2$, $v_3$,... denote the velocities at $B$, $C$, $D$,... respectively. Then if no velocity were destroyed in passing from plane to plane we should have the following equations by Arts. 28 and 40,

$$v_1^2 = 2gh_1,$$
$$v_2^2 = v_1^2 + 2gh_2,$$
$$v_3^2 = v_2^2 + 2gh_3,$$

and so on.

Hence, by addition, supposing there are $n$ planes,

$$v_n^2 = 2g(h_1 + h_2 + h_3 + \dots + h_n) = 2gh,$$

where $h$ denotes the whole vertical height.

We must now consider what velocity is lost in passing from plane to plane. We assume that the body is *inelastic.*

Let us suppose the angle between any plane and the next plane produced to be the same; denote it by $\alpha$.

Resolve the velocity at $B$ along $BC$ and at right angles to $BC$; the former component is $v_1 \cos\alpha$, and the latter is $v_1 \sin\alpha$: the former will be the velocity at the beginning of the motion down $BC$, for the latter is destroyed by the plane $BC$. See Art. 117. Hence instead of $v_2^2 = v_1^2 + 2gh_2$ we have $v_2^2 = v_1^2 \cos^2\alpha + 2gh_2$.

Similarly we obtain $v_3^2 = v_2^2 \cos^2\alpha + 2gh_3$, and so on.

Hence, by addition,

$$v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2 = (v_1^2 + v_2^2 + v_3^2 + \ldots + v_{n-1}^2)\cos^2\alpha + 2gh;$$

therefore $\quad v_n^2 = 2gh - \Sigma$, where $\Sigma$ stands for

$$\sin^2\alpha\,(v_1^2 + v_2^2 + v_3^2 + \ldots + v_{n-1}^2).$$

We shall now shew that $\Sigma$ vanishes when the number of planes is made indefinitely great.

It is obvious that $\Sigma$ is less than $(n-1)\,v_{n-1}^2 \sin^2\alpha$. Let $\beta$ be the angle between the first plane and the last plane produced; then $\beta = (n-1)\,\alpha$. Hence $\Sigma$ is less than $\frac{\beta}{\alpha} v_{n-1}^2 \sin^2\alpha$, that is less than $\beta \sin\alpha \,.\, \frac{\sin\alpha}{\alpha} v_{n-1}^2$. Now we know from Trigonometry that $\frac{\sin\alpha}{\alpha}$ is less than unity, so that $\Sigma$ is less than $\beta \sin\alpha\, v_{n-1}^2$. Hence by making $\alpha$ small enough we can make $\Sigma$ less than any assigned quantity. Thus ultimately $\Sigma = 0$.

Hence $v_n^2 = 2gh$, which was to be shewn.

148. In a similar way we may obtain the following result: *if a body start with the velocity* u *and move in contact with a smooth curve in a vertical plane the velocity when the body has risen through the vertical height* h *is* $\sqrt{(u^2 - 2gh)}$.

149. The demonstration in Art. 147 is that which is usually given in elementary works; when the student has sufficient mathematical knowledge to read more elaborate treatises on Dynamics, he will find that the result can be obtained in a more satisfactory manner, and without assuming that the body is inelastic.

150. Let one end of a fine string be fastened to a fixed point; and let a heavy particle be attached to the other end. In the position of equilibrium the string will be vertical. Let the particle be displaced from this position, the string being kept stretched, and then allowed to move. The particle will oscillate backwards and forwards describing an arc of a circle; the arc continually diminishes owing to the resistance of the air, until the particle comes to rest. The system is called a *simple pendulum.*

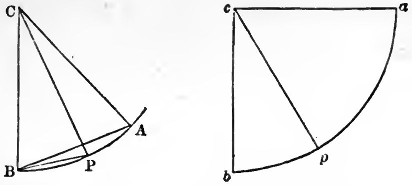
Now it is a matter of great interest to determine the time in which the particle describes an arc, or rather the time in which it would describe an arc neglecting the resistance of the air. This we shall consider in the next Article. The investigation is somewhat complex; but it deserves attention because the student will find hereafter when he has the Differential and Integral Calculus at his command, that although some of the steps may be abbreviated, yet the process cannot be essentially improved.

We assume as obvious that the motion is exactly the same whether the particle is compelled to describe an arc of a circle by means of a string or a fine straight wire in the manner just explained, or whether the particle moves in a fine tube in the manner of Art. 147.

151. *To find the time of descent of a particle moving in a circle under the action of gravity.*

Let $APB$ be an arc of a circle of which $C$ is the centre, and $B$ vertically under $C$; let a particle start from $A$ and move along the curve to $B$: required the time of the motion.

Let $P$ be any point in the arc; let the radius $CP=r$; let the angle $BCA=\alpha$, and the angle $BCP=\theta$. Let $\pi$ denote as usual the ratio of the circumference of a circle to its diameter.




When the particle is at $P$ the square of its velocity, by Art. 147,

$$=2g(r\cos\theta - r\cos\alpha)=4gr\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right).$$

Now $$AB=2r\sin\frac{\alpha}{2}, \quad PB=2r\sin\frac{\theta}{2}.$$

Assume $\sin\frac{\theta}{2}=\sin\frac{\alpha}{2}\cos\phi$, so that $\cos\phi$ denotes the ratio of $BP$ to $BA$, and thus as the particle moves from $A$ to $B$ the angle $\phi$ changes from 0 to $\frac{\pi}{2}$.

Describe a quarter of a circle $apb$ with radius $r$; let $c$ be the centre, and let the angle $acp=\phi$. Then we have in fact to find the time in which the point $p$ describes the arc $ab$. We must first determine the velocity of $p$.

Suppose $P$ to move to a new position $P'$, such that the angle $BCP'=\theta'$; and let $p$ move to a corresponding new position $p'$, such that the angle $acp'=\phi'$. Then the velocity of $P$ is to the velocity of $p$ as the chord $PP'$ is to the chord $pp'$ when these *chords are indefinitely small.* But we have

$$\sin\frac{\theta}{2}=\sin\frac{\alpha}{2}\cos\phi, \quad \sin\frac{\theta'}{2}=\sin\frac{\alpha}{2}\cos\phi';$$

therefore $$\sin\frac{\theta}{2}-\sin\frac{\theta'}{2}=\sin\frac{\alpha}{2}(\cos\phi-\cos\phi'),$$

that is, $$\sin\frac{\theta-\theta'}{4}\cos\frac{\theta+\theta'}{4}=\sin\frac{\alpha}{2}\sin\frac{\phi'-\phi}{2}\sin\frac{\phi'+\phi}{2};$$

therefore
$$\frac{\sin\frac{\phi'-\phi}{2}}{\sin\frac{\theta-\theta'}{4}}=\frac{\cos\frac{\theta+\theta'}{4}}{\sin\frac{a}{2}\sin\frac{\phi+\phi'}{2}}.$$

Now
$$\frac{\text{chord } pp'}{\text{chord } PP'}=\frac{2r\sin\frac{\phi'-\phi}{2}}{2r\sin\frac{\theta-\theta'}{2}}$$

$$=\frac{\sin\frac{\phi'-\phi}{2}}{2\sin\frac{\theta-\theta'}{4}\cos\frac{\theta-\theta'}{4}}=\frac{\cos\frac{\theta+\theta'}{4}}{2\sin\frac{a}{2}\sin\frac{\phi+\phi'}{2}\cos\frac{\theta-\theta'}{4}}.$$

Hence when the chords are indefinitely small this ratio

becomes
$$\frac{\cos\frac{\theta}{2}}{2\sin\frac{a}{2}\sin\phi}.$$

The velocity of $P$

$$=2\sqrt{gr}\sqrt{\left(\sin^2\frac{a}{2}-\sin^2\frac{a}{2}\cos^2\phi\right)}=2\sqrt{gr}\sin\frac{a}{2}\sin\phi\,;$$

therefore the velocity of $p$

$$=\frac{2\sqrt{gr}\sin\frac{a}{2}\sin\phi\cos\frac{\theta}{2}}{2\sin\frac{a}{2}\sin\phi}=\sqrt{gr}\cos\frac{\theta}{2}$$

$$=\sqrt{gr}\sqrt{\left(1-\sin^2\frac{\theta}{2}\right)}=\sqrt{gr}\sqrt{\left(1-\sin^2\frac{a}{2}\cos^2\phi\right)}.$$

If we assume $a$ to be *very small*, $\sin^2\frac{a}{2}$ is extremely small,

and thus the velocity of $p$ is very nearly $\sqrt{gr}$. Hence the required time is very nearly $\dfrac{\frac{\pi}{2}r}{\sqrt{gr}}$, that is, $\dfrac{\pi}{2}\sqrt{\dfrac{r}{g}}$.

152. The particle will take the same time in rising from the lowest point of the arc to a height equal to that from which it descended: see Art. 148. Thus the whole time of moving from the extreme position on one side of the vertical to the extreme position on the other side is $\pi\sqrt{\dfrac{r}{g}}$: this extent of motion is called an *oscillation*.

If we wish to find the length of a simple pendulum which will oscillate once in a second we put $\pi\sqrt{\dfrac{r}{g}}=1$; thus $r=\dfrac{g}{\pi^2}$. Thus taking $g=32$ feet, the length of the seconds' pendulum is about $\dfrac{32}{(3{\cdot}14)^2}$ *feet;* this will be found to be about 39 *inches.* The British standard of length is connected with the length of the seconds' pendulum by an Act of Parliament, which defines the inch to be such that the length of a simple pendulum which oscillates in a second in the latitude of London shall be 39·1393 inches.

153. It will be seen that the investigation in Art. 151 is *exact* up to the point at which we find that the velocity of $p$ is $\sqrt{gr}\sqrt{\left(1-\sin^2\dfrac{a}{2}\cos^2\phi\right)}$, and then we take an approximate value of this expression instead of the exact value. It is not difficult to make a closer approximation, assuming still that $a$ is small.

Suppose $n$ a large number, and let $n\beta=\dfrac{\pi}{2}$ so that $\beta$ is a very small angle. Let $\phi=m\beta$, and assume that while the angle $acp$ changes from $m\beta$ to $(m+1)\beta$ we may consider the velocity of $p$ to be always $\sqrt{gr}\sqrt{\left(1-\sin^2\dfrac{a}{2}\cos^2 m\beta\right)}$.

Then the time of describing this portion of the quadrant

$$=\frac{r\beta}{\sqrt{rg}\sqrt{\left(1-\sin^2\frac{a}{2}\cos^2 m\beta\right)}}=\frac{\beta\sqrt{r}}{\sqrt{g}}\left(1-\sin^2\frac{a}{2}\cos^2 m\beta\right)^{-\frac{1}{2}}$$

$=\frac{\beta\sqrt{r}}{\sqrt{g}}\left(1+\frac{1}{2}\sin^2\frac{a}{2}\cos^2 m\beta\right)$ nearly, by the Binomial Theorem. Then we have to find the sum of the values of this expression for all values of $m$ from $0$ to $n-1$ inclusive. Thus the time required is $\frac{\sqrt{r}}{\sqrt{g}}\left(n\beta+\frac{\beta}{2}S\sin^2\frac{a}{2}\right)$, where $S$ stands for $1+\cos^2\beta+\cos^2 2\beta+\ldots\ldots+\cos^2(n-1)\beta$.

But since $\sin m\beta=\cos\left(\frac{\pi}{2}-m\beta\right)=\cos(n-m)\beta$, we have

$$S=1+\sin^2(n-1)\beta+\sin^2(n-2)\beta+\ldots\ldots+\sin^2\beta,$$

and also

$$S=\quad\cos^2(n-1)\beta+\cos^2(n-2)\beta+\ldots\ldots+\cos^2\beta+1\,;$$

thus by addition $2S=n+1$. Also $n\beta=\frac{\pi}{2}$.

Therefore the required time

$$=\sqrt{\frac{r}{g}}\left(\frac{\pi}{2}+\frac{\pi}{4n}\cdot\frac{n+1}{2}\sin^2\frac{a}{2}\right)=\frac{\pi}{2}\sqrt{\frac{r}{g}}\left(1+\frac{n+1}{4n}\sin^2\frac{a}{2}\right).$$

Let $n$ increase indefinitely: then we obtain finally for the required time

$$\frac{\pi}{2}\sqrt{\frac{r}{g}}\left(1+\frac{1}{4}\sin^2\frac{a}{2}\right).$$

154. Thus in Art. 151 we have found an approximate value of the time of motion; and in Art. 153 a still closer approximation: the smaller the value of $a$ is the less will be the error in taking these approximations for the exact time. By the aid of the higher parts of mathematics we can find an expression, in the form of a series, which will determine the time, as nearly as we please, whatever be the value of $a$.

## EXAMPLES. XIII.

1. A particle slides down an arc of a circle to the lowest point: find the velocity at the lowest point if the angle described round the centre is $60^0$.

2. If the length of the seconds' pendulum be 39·1393 inches find the value of $g$ to three places of decimals.

3. A pendulum which oscillates in a second at one place is carried to another place where it makes 120 more oscillations in a day: compare the force of gravity at the latter place with that at the former.

4. Suppose that $l$ is the length of the seconds' pendulum, and that the lengths of two other pendulums are $l-c$ and $l+c$ respectively, where $c$ is very small: shew that the sum of the number of oscillations of these two pendulums in a day is very nearly $2 \times 24 \times 60 \times 60 \left(1 + \frac{3c^2}{8l^2}\right)$.

5. A pendulum is found to make $p$ oscillations at one place in the same time as it makes $q$ oscillations at another. Shew that if a string hanging vertically can just support $n$ cubic inches of a given substance at the former place it will just support $\frac{np^2}{q^2}$ cubic inches at the latter place.

6. A seconds' pendulum hangs against the smooth face of an inclined wall and swings in its plane: find the time of a small oscillation.

7. A seconds' pendulum is carried to the top of a mountain $m$ miles high: assuming that the force of gravity varies inversely as the square of the distance from the centre of the earth, find the time of a small oscillation.

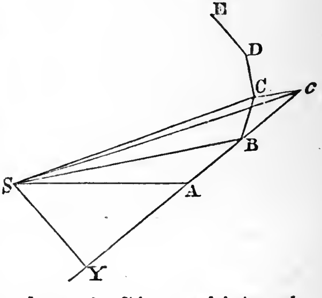
8. Shew that the length of a pendulum which will make a small oscillation in one second at the top of a mountain $m$ miles high is $\left(\frac{4000}{4000+m}\right)^2 l$, where $l$ is the length of the seconds' pendulum at the surface of the earth.

## XIV. *Uniform motion in a Circle.*

155. If the direction of a force always passes through a fixed point the force is called a *central force;* and the fixed point is called the *centre of force.*

In the present Chapter and the next two Chapters we shall be occupied with cases of central forces: we begin with some propositions due to Newton which are contained in the next five Articles.

156. *When a body moves under the action of a central force the areas described by the radius drawn to the centre of force are in one plane and are proportional to the times of describing them.*




Let $S$ be the centre of force; and suppose a body acted on by no force to describe the straight line $AB$, with uniform velocity, in a given interval of time. In another equal interval, if no force acted, the body would describe $Bc$ equal to $AB$, in $AB$ produced, so that the equal areas $ASB$ and $BSc$ would be described by the radius drawn to $S$ in equal intervals.

But when the body arrives at $B$ let a force tending to $S$ act on it by an impulse, and cause it to proceed in the direction $BC$ instead of $Bc$; then if $C$ be the position of the body at the end of the second interval $Cc$ is parallel to $BS$: see Art. 47. Join $SC$; then the triangle $BSC$ is equal to the triangle $BSc$, by Euclid, I. 37; therefore the triangle $BSC$ is equal to the triangle $ASB$, and the two triangles are in the same plane.

In like manner if impulses tending to $S$ act on the body at $C$, $D$, $E$,...causing the body to describe in successive equal intervals the straight lines $CD$, $DE$,..., the triangles $CSD$, $DSE$,... are all equal to the triangle $ASB$, and are in the same plane with it.

Thus equal areas are described in equal intervals, and the sum of any number of areas is proportional to the time of description.

Now let the number of triangles be indefinitely increased, and the base of each indefinitely diminished; then the boundary $ABCDE$...will ultimately become a curve, and the series of impulses will become a continuous central force by the action of which the body is made to describe the curve. And the areas described being always proportional to the times will be so also in this case.

157. The proposition of the preceding Article is true also if $S$ be a point which instead of being fixed moves uniformly in a straight line. For by the fifth Corollary in Art. 142 the relative motion is the same whether the plane in which the curve is described be at rest or be moving with the body and the curve and the point $S$ uniformly in a straight line.

158. *If* v *be the velocity of the body at any point* A, *and* p *the perpendicular from* S *on the tangent at that point, the area described in the time* $\mathrm{t} = \frac{1}{2}\mathrm{ptv}$.

Draw $SY$ perpendicular to $AB$. Let $t$ be divided into $n$ equal intervals, and let $AB$ be the space described in the first interval, the force at $S$ being supposed to act by impulses at the end of each interval.

Then the polygonal area which is described in the time $t$ $= n$ times the triangle $SAB = n\frac{1}{2}SY.\frac{t}{n}.v = \frac{1}{2}SY.t.v.$

In the limit the straight line $AB$, which is the direction of the velocity at $A$, becomes the tangent to the curve at $A$; and the curvilinear area described in the time $t = \frac{1}{2}ptv$.

Thus the area described in a unit of time is $\frac{1}{2}pv$; it is usual to denote twice the area described in a unit of time by $h$: therefore $h=pv$, and $v=\frac{h}{p}$.

159. *If a body move in one plane so that the areas described by the radius drawn to a fixed point are proportional to the times of describing them the body is acted on by a force tending to that point.*

Let $S$ be the fixed point about which areas proportional to the times are described, and suppose a body acted on by no force to describe the straight line $AB$ with uniform velocity in a given interval of time. In another equal interval if no force acted the body would describe $Bc$ equal to $AB$, in $AB$ produced: so that the triangles $ASB$ and $BSc$ would be equal. But when the body arrives at $B$ let a force act on it by an impulse which causes it to describe $BC$ in the second interval, such that the triangle $SBC$ is equal to the triangle $ASB$, and in the same plane.

Then the triangle $BSC$ is equal to the triangle $BSc$, and therefore $Cc$ is parallel to $SB$, by Euclid, I. 39: therefore the impulse at $B$ is in the direction $BS$: see Art. 47.

In like manner if impulses act on the body at $C, D, E,\ldots$ causing the body to describe in successive equal intervals the straight lines $CD, DE,\ldots$ so that the triangles $CSD$, $DSE,\ldots$are all equal to the triangle $ASB$, and are in the same plane with it, then all the impulses tend to $S$.

Hence if any polygonal areas be described proportional to the times of describing them, the impulses at the angular points all tend to $S$.

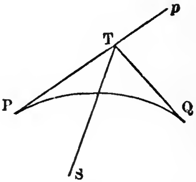
Now let the number of triangles be indefinitely increased, and the base of each indefinitely diminished; then the boundary $ABCDE\ldots$ will ultimately become a curve, and the series of impulses will become a continuous force by the action of which the body is made to describe the curve: and the force always tends to $S$.

160. The proposition of the preceding Article is true also if $S$ be a point which instead of being fixed moves uniformly in a straight line; see Art. 157.

161. We have already observed in Art. 49 that the principle called the Parallelogram of Velocities gives rise to applications similar to those deduced from the Parallelogram of Forces in Statics; some illustrations of this remark will occur as we proceed, one of great interest being given in the next Article.

162. *The direction of the resultant action of a central force on a body while it describes an arc of a curve is the straight line which joins the intersection of the tangents at the extremities of the arc with the centre of force.*

Let $PQ$ be an arc of a curve described by a body under the action of a centre of force at $S$. Let $PT$, $QT$ be the tangents at $P$ and $Q$ respectively. Suppose the body to move from $P$ to $Q$. Produce $PT$ to any point $p$.

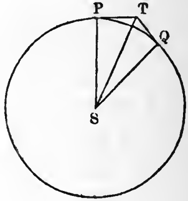



The resultant action of the central force during the motion changes the direction of the velocity from $Tp$ to $TQ$; and thus the direction of the resultant action must pass through $T$. But the direction of the action of the central force passes through $S$ at every instant, and therefore the direction of the resultant action must pass through $S$. Thus $TS$ must be the direction of the resultant action.

This proposition has been given on account of its simplicity and interest; but it is not absolutely necessary for the purposes of the present work, for it will be found that so much of the result as we may hereafter require will present itself naturally in the course of our investigations. See Arts. 163 and 175.

163. *If a body describes a circle of radius* r, *with uniform velocity* v, *the body is acted on by a force tending to the centre of the circle, the acceleration of which is* $\frac{\mathrm{v}^2}{\mathrm{r}}$.

Since the body moves with uniform velocity the arc described in any time is proportional to the time. Hence, by Euclid, VI. 33, the area described in any time by the radius drawn to the centre is proportional to the time. Therefore, by Art. 159, the body is acted on by a force always tending to the centre of the circle.




Let $PQ$ be an arc of the circle, $S$ the centre, $PT$ and $QT$ the tangents at $P$ and $Q$ respectively.

Let the angle $PSQ$ be expressed in *Circular Measure* and denoted by $2\phi$, and let $u$ denote the velocity communicated to the body by the action of the central force while the body moves from $P$ to $Q$. Then the velocity $v$ along $TQ$ is the resultant of $v$ along $PT$ and of $u$ communicated by the central force. Hence as in Art. 33 of the Statics the direction of $u$ makes equal angles with $TP$ and $TQ$; and must in fact coincide with $TS$. Hence, as in Art. 38 of the Statics,

$$\frac{u}{v}=\frac{\sin PTQ}{\sin PTS}=\frac{\sin PSQ}{\cos PST}=\frac{\sin 2\phi}{\cos\phi}=2\sin\phi.$$

Let $t$ denote the time in which the body moves from $P$ to $Q$, and let $f$ denote the accelerating effect of the central force. Then if we suppose $Q$ very near to $P$ we have $u=ft$, because during a very small change of position of the body the force may be considered as constant in magnitude and in direction. Hence $ft=2v\sin\phi$. But since the velocity is uniform we have $2r\phi=vt$, as in Art. 3; for $2r\phi$ is the length of the arc $PQ$. From these two equations we obtain

$$f=\frac{2v^2\sin\phi}{2r\phi}=\frac{v^2}{r}\cdot\frac{\sin\phi}{\phi}.$$

But when $\phi$ is indefinitely small we have by Trigonometry, $\frac{\sin\phi}{\phi} = 1$ : thus $f = \frac{v^2}{r}$.

164. The preceding investigation of the value of the central force should be carefully studied.

In the first part of the investigation we obtain the exact direction and amount of the velocity $u$ communicated by the central force while the body describes a given arc.

In the second part we have to use the *method of limits*, that is, we write down equations which are true in the limit, namely when the arc described is supposed indefinitely small.

The reasoning in the second part of the investigation might be given more fully in the following manner. Let $f_1$ be the greatest value of the acceleration, and $f_2$ the least, while the body describes the arc $PQ$. Then $u$ cannot be so large as $f_1t$, and cannot be so small as $f_2t\cos 2\phi$ : for $f_1t$ would be the velocity generated if the force always acted in the same direction, and had its greatest possible value; and $f_2t$ would be the velocity generated if the force always acted in the same direction and had its least possible value; and as $2\phi$ is the angle between the extreme directions of the force, if the force always had its least value the velocity generated would be greater than $f_2t\cos 2\phi$. Hence $u$ lies between $f_1t$ and $f_2t\cos 2\phi$; that is $2v\sin\phi$ lies between $f_1t$ and $f_2t\cos 2\phi$; therefore $\frac{v^2}{r}\frac{\sin\phi}{\phi}$ lies between $f_1$ and $f_2\cos 2\phi$. Since this is always true it is true at the limit. Now the limit of $\frac{v^2}{r}\frac{\sin\phi}{\phi}$ is $\frac{v^2}{r}$; and the limit of $f_1$ and of $f_2\cos 2\phi$ is $f$. Thus $\frac{v^2}{r} = f$.

165. It will be seen that we demonstrate that the acceleration has the same value at every point of the circle: this might have been anticipated but we did not assume it.

In the manner thus exemplified we may in similar cases develop the reasoning, so as to render it more rigorous in form : the student will have no difficulty in supplying such a development for himself on other occasions if required.

166. We have thus demonstrated the following result: *if a body of mass* m *describes a circle of radius* r, *with uniform velocity* v, *then whatever be the forces acting on the body their resultant tends to the centre of the circle, and is equal to* $\frac{mv^2}{r}$. No single fact in the whole range of Dynamics is of greater importance than this, and the student should regard it with earnest attention.

167. For example, suppose a body of mass $m$ fastened to one end of a string, and the other end of the string fastened to a fixed point in a smooth horizontal table. Let the body be started in such a manner as to describe a circle with uniform velocity, $v$, on the table round the fixed point, the string forming the radius, $r$, of the circle. The forces acting on the body are its weight, the resistance of the table, and the tension of the string. The weight and the resistance act vertically and balance each other. The tension of the string acts horizontally, and its value must be equal to $\frac{mv^2}{r}$.

This may be verified experimentally. Instead of fastening the string to a fixed point in the plane, let the string be prolonged and pass through a hole in the position of the fixed point, and have a body fastened to its end. Let $m'g$ denote the weight of this body; then if it remains at rest we shall find that $m'g = \frac{mv^2}{r}$.

168. For another example we will take the *conical pendulum*. One end of a fine string is fixed; to the other end a particle is fastened. The particle is set in motion in such a manner as to describe a horizontal circle with uniform velocity: thus the string traces out the surface of a right cone, from which the name *conical pendulum* is derived.

Let $mg$ be the weight of the particle, $v$ its velocity, $T$ the tension of the string, $l$ its length, $\alpha$ the inclination of the string to the vertical.

The vertical forces acting on the particle are its weight and the resolved part of the tension; these must be in

equilibrium, so that

$$T \cos a = mg.$$

The only horizontal force is the resolved part of the tension; therefore by Art. 166

$$T \sin a = \frac{mv^2}{r};$$

also

$$r = l \sin a.$$

Hence $\tan a = \frac{v^2}{rg} = \frac{v^2}{lg \sin a}$, or $\frac{\sin^2 a}{\cos a} = \frac{v^2}{lg}$.

This relation then must hold between $v$, $l$, and $a$, in order that the supposed motion may take place.

169. For another example, we will take the case of the moon moving round the earth; this example is of special interest as being that by which Newton tested his law of gravitation.

It appears from observation that the moon moves nearly in a circle, with uniform velocity, round the earth as centre. Let $v$ denote the moon's velocity and $r$ the distance of the moon from the earth's centre. Hence the acceleration on the moon is $\frac{v^2}{r}$.

Now Newton conjectured that this acceleration was owing to the earth's attraction, that the fall of heavy bodies to the earth's surface was due to the same cause, and that the force of the earth's attraction varied inversely as the square of the distance.

Let $a$ denote the earth's radius; then, since $g$ denotes the acceleration produced by the earth's attraction at the surface of the earth, the acceleration produced at the distance of the moon will be $\frac{ga^2}{r^2}$. Hence, if Newton's conjecture be true, we must have

$$\frac{ga^2}{r^2} = \frac{v^2}{r}.$$

It is said that when Newton first tried the calculation the result was not satisfactory; the value of $a$ not being known at that time with sufficient accuracy; but at a subsequent

period, having obtained a more accurate value of $a$, he returned to the calculation and obtained the desired agreement.

But there does not appear to be decisive authority for the statement: see Rigaud's *Historical Essay on the first publication of Sir Isaac Newton's Principia*, page 6.

The student can easily verify the result approximately, taking the following facts as given by observation:

$a = 4000$ miles $= 4000 \times 5280$ feet,

$r = 60a$,

$$v = \frac{2\pi r}{\text{Time of moon's revolution}} = \frac{2\pi r}{27\frac{1}{4} \times 24 \times 60 \times 60}.$$

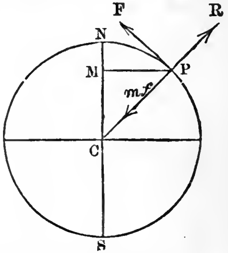
The preceding investigation is sufficient to give a general idea of the circumstances of the motion of the moon; but many additional considerations enter into an exact discussion of the subject. The moon does not move accurately in a circle round the earth, nor with quite uniform velocity; and the sun exerts an influence on the motion: but the investigation of these points is altogether beyond the student at present.

170. Suppose a body to describe a circle of radius $r$ with uniform velocity $v$. Then as the circumference is $2\pi r$, the body moves once round in the time $\frac{2\pi r}{v}$. This is called the *periodic time;* and generally when a body describes any closed curve the time of going once round is called the *periodic time.*

171. When a body is describing a circle with uniform velocity the straight line drawn from the body to the centre describes equal angles in equal times. The rate at which angles are described is called the *angular velocity* of the radius. Thus with the notation of the preceding Article the periodic time is $\frac{2\pi r}{v}$; and in this time the angle $2\pi$ is traced out: thus the angular velocity is $2\pi \div \frac{2\pi r}{v}$, that is $\frac{v}{r}$.

The velocity $v$ is called the *linear* velocity when it is necessary to distinguish it from the *angular* velocity, which is equal to the linear velocity divided by the radius.

172. *To find the pressure which a body on the surface of the earth at any point exerts on the earth, supposing the earth to be a sphere of uniform density.*




Let $NCS$ represent the axis on which the earth turns; suppose a body at $P$ resting on the earth and turning round with it. Let $C$ be the centre of the earth, and suppose $PC$ to make an angle $\theta$ with the plane of the equator, so that $\theta$ is the *latitude* of the body, and $PCN$ is the complement of $\theta$.

Let $m$ be the mass of the body. The body is acted on by the following forces: the attraction of the earth towards its centre, which we will denote by $mf$, the resistance of the earth along the radius $CP$, which we will denote by $R$, the friction in the plane $NCP$ along the tangent at $P$, which we will denote by $F$. Draw $PM$ perpendicular to $CN$; then as $P$ revolves with the earth it describes a circle of radius $PM$ with uniform velocity. Let $PM=r$, and let $v$ denote the velocity of $P$, and $a$ the earth's radius. Then the forces which act on the body at $P$ must have their resultant along $PM$, and equal to $\frac{mv^2}{r}$. Therefore

$$(mf-R).\cos\theta+F\sin\theta=\frac{mv^2}{r},\quad (mf-R)\sin\theta-F\cos\theta=0.$$

Multiply the first of these equations by $\cos\theta$, and the second by $\sin\theta$, and add; thus

$$mf-R=\frac{mv^2}{r}\cos\theta,$$

that is
$$R=mf-\frac{mv^2}{r}\cos\theta.$$

Let $\omega$ denote the angular velocity of the earth, and therefore of $P$; then $v=r\omega$: also $r=a\cos\theta$.

Thus $$R=mf-ma\omega^2\cos^2\theta.$$

Similarly we find $F=ma\omega^2\sin\theta\cos\theta$.

The resultant pressure on the earth is equal and opposite to the resultant of $R$ and $F$. Denote it by $G$. Then the direction of $G$ makes with the radius $CP$ an angle the tangent of which is $\frac{F}{R}$, that is $\frac{ma\omega^2\sin\theta\cos\theta}{mf-ma\omega^2\cos^2\theta}$.

And
$$\begin{aligned} G^2 &= m^2\{(f-a\omega^2\cos^2\theta)^2+a^2\omega^4\sin^2\theta\cos^2\theta\} \\ &= m^2\{f^2-2af\omega^2\cos^2\theta+a^2\omega^4\cos^2\theta\} \\ &= m^2\{f^2-(2af\omega^2-a^2\omega^4)\cos^2\theta\}. \end{aligned}$$

The quantity $G$ is what we have hitherto denoted by $mg$; we see that it is not the same for all places on the earth's surface. We shall proceed to some numerical estimate.

The earth revolves once in twenty-four hours; thus $\omega=\frac{2\pi}{24\times 60\times 60}$: also $a=4000\times 5280$. Here we take as usual a second as the unit of time, and a foot as the unit of length.

We shall find that $a\omega^2=\frac{\pi^2}{90}$ nearly; the square of this, that is $a^2\omega^4$, we shall neglect in comparison with $2af\omega^2$. Thus approximately $g^2=f^2\left(1-\frac{2a\omega^2}{f}\cos^2\theta\right)$, and therefore approximately $g=f\left(1-\frac{a\omega^2}{f}\cos^2\theta\right)$. We know that $f=32$ nearly, and thus $\frac{a\omega^2}{f}=\frac{1}{289}$ nearly; so that $g=f\left(1-\frac{\cos^2\theta}{289}\right)$ nearly.

We have assumed that the earth is a sphere, and that the attraction which it exerts on a body placed at any point on the surface is directed towards the centre; but these assumptions are not strictly accurate, so that the result must not be considered absolutely true.

## EXAMPLES. XIV.

1. Find the force towards the centre required to make a body move uniformly in a circle whose radius is 5 feet, with such a velocity as to complete a revolution in 5 seconds.

2. A stone of one lb. weight is whirled round horizontally by a string two yards long having one end fixed: find the time of revolution when the tension of the string is 3 lbs.

3. A body weighing $P$ lbs. is at one end of a string, and a body weighing $Q$ lbs. at the other end; the system is in motion on a smooth horizontal table, $P$ and $Q$ describing circles with uniform velocities: determine the position of the point in the string which does not move.

4. A string $l$ feet long can just support a weight of $P$ lbs. without breaking; one end of the string is fixed to a point on a smooth horizontal table; a weight of $Q$ lbs. is fastened to the other end and describes a circle with uniform velocity round the fixed point as centre: determine the greatest velocity which can be given to the weight of $Q$ lbs. so as not to break the string.

5. One end of a string is fixed; to the other end a particle is attached which describes a horizontal circle with uniform velocity so that the string is always inclined at an angle of $60^0$ to the vertical: shew that the velocity of the particle is that which would be acquired in falling freely from rest through a space equal to three-fourths of the length of the string.

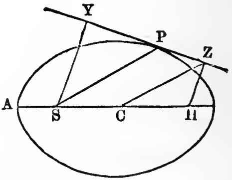
## XV. *Motion in a Conic Section round a focus.*

173. The cases of motion which we shall discuss in the present Chapter are of great interest on account of the application of them to the earth and the planets which describe ellipses round the sun in a focus.

In the present Chapter and the next Chapter we shall consider the action of a force on a *given* body, so that we shall be occupied only with the influence of the force on the *velocity* of the body: see Arts. 14, 45.

174. *If a body describes an ellipse under the action of a force in a focus, the velocity at any point can be resolved into two components, both constant in magnitude, one perpendicular to the major axis of the ellipse, and the other at right angles to the radius drawn from the body to the focus.*

Let $S$ be the focus which is the centre of force, $H$ the other focus, $P$ any point on the ellipse, $SY$ and $HZ$ perpendiculars from $S$ and $H$ on the tangent at $P$. Let $C$ be the centre of the ellipse, $A$ one end of the major axis.



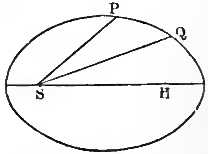

By Art. 158 the velocity at $P$ varies inversely as $SY$, and therefore directly as $HZ$; for $SY \times HZ$ is constant, by a property of the ellipse, being equal to the square on half the minor axis of the ellipse. Thus $HZ$ may be taken to represent the velocity in magnitude, and it is at right angles to the velocity in direction. Now a velocity represented by $HZ$ may be resolved into two represented by $HC$ and $CZ$. And by the nature of the ellipse $CZ$ is parallel to $SP$ and equal to $CA$.

Hence a velocity represented by $HZ$ in magnitude, and at right angles to $HZ$ in direction, may be resolved into two, one represented by $CA$ in magnitude and at right

angles to $SP$ in direction, and the other represented by $HC$ in magnitude and perpendicular to $HS$ in direction.

It is convenient to have expressions for the magnitudes of these component velocities. Let $CA=a$, let $b$ denote half the minor axis, and let $e$ be the excentricity of the ellipse. Let $h$ represent twice the area described by the radius $SP$ in a unit of time; then the velocity at $P=\frac{h.HZ}{SY\times HZ}=\frac{h.HZ}{b^2}$. Therefore the component at right angles to $SP$ is $\frac{h.CA}{b^2}$, that is $\frac{ha}{b^2}$; and the component perpendicular to $HS$ is $\frac{h.CH}{b^2}$, that is $\frac{hae}{b^2}$.

175. *A body describes an ellipse under the action of a force in a focus: find the law of force.*




Let $S$ be the focus which is the centre of force; let $P$ and $Q$ be any two points on the ellipse; and suppose the body to move from $P$ to $Q$.

Resolve the velocity at $P$ into two components one at right angles to $SP$, and the other perpendicular to $HS$; denote these by $v_1$ and $v_2$ respectively. When the body arrives at $Q$ its velocity is composed of $v_1$ and $v_2$ parallel to their directions at $P$, and the velocity generated by the action of the central force during the motion, which we will denote by $u$. But by Art. 174 the velocity at $Q$ can be resolved into $v_1$ at right angles to $SQ$, and $v_2$ perpendicular to $SH$.

Hence it follows that $v_1$ at right angles to $SP$ together with $u$ in its own direction have for their resultant $v_1$ at right angles to $SQ$. Hence, as in Art. 33 of the Statics, the direction of $u$ makes equal angles with the straight lines at right angles to $SP$ and $SQ$, and therefore with $SP$ and $SQ$. And, by Art. 38 of the Statics,

$$u=2v_1 \sin \frac{1}{2} PSQ.$$

Let $SP=r$, and $PSQ=2\phi$. Let $t$ denote the time in which the body moves from $P$ to $Q$; and let $f$ denote the accelerating effect of the force. Then if we suppose $Q$ very near to $P$ so that $t$ is very small, we have $u=ft$; hence $ft=2v_1 \sin\phi$. The area described in passing from $P$ to $Q=\frac{1}{2}ht$ by Art. 158; and this area may be taken to be $=\frac{1}{2}r^2 \sin 2\phi$, for it may be considered ultimately as a triangle.

Thus $\frac{1}{2}ht=\frac{1}{2}r^2 \sin 2\phi$.

Therefore $$fr^2 \sin 2\phi=2hv_1 \sin\phi;$$

therefore $$f=\frac{2hv_1}{2r^2 \cos\phi}=\frac{hv_1}{r^2} \text{ ultimately,}$$

when $\phi$ is made indefinitely small.

This shews that the force *varies inversely as the square of the distance.*

It is usual to denote the constant $hv_1$ by $\mu$; thus $\mu=h \times \frac{ha}{b^2}=\frac{h^2a}{b^2}$. The quantity $\mu$ is called the *absolute force.*

176. In the preceding investigation it was shewn that the direction of the velocity $u$ communicated by the central force while the body moves from $P$ to $Q$ *bisects the angle PSQ.* But we know by Art. 162 that this direction is that of the straight line which joins $S$ with the intersection of the tangents at $P$ and $Q$. Thus our dynamical investigation suggests that in an ellipse the two tangents from an external point subtend equal angles at a focus; and this is a known property of the ellipse.

177. *A body describes an ellipse under the action of a force in a focus: required to determine the periodic time.*

Let $a$ and $b$ denote the semiaxes of the ellipse; and $h$ twice the area described by the radius in the unit of time. By Art. 156 the periodic time

$$=\frac{\text{twice the area of the ellipse}}{h}.$$

Now it is known that the area of the ellipse is $\pi ab$, and by Art. 175 we have $h=\frac{b\sqrt{\mu}}{\sqrt{a}}$. Hence the periodic time

$$=2\pi ab\div\frac{b\sqrt{\mu}}{\sqrt{a}}=\frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}.$$

178. We can now apply the results obtained to the motions of the earth and the planets round the sun. There are certain facts connected with these motions which were discovered in the seventeenth century by the diligence of Kepler, a famous German astronomer, and which are justly called *Kepler's Laws.* These laws are the following:

(1) The planets describe ellipses round the sun in a focus.

(2) The radius drawn from a planet to the sun describes in any time an area proportional to the time.

(3) The squares of the periodic times are proportional to the cubes of the major axes of the orbits.

From the second law it follows, by Art. 159, that each planet is acted on by a force tending to the sun.

From the first law it follows, by Art. 175, that the force on each planet varies inversely as the square of the distance.

From the third law an important inference can be drawn, as we will now shew. Let $a$ be the semiaxis major of the ellipse described by one planet, $\mu$ the absolute force, $T$ the periodic time; let $a'$, $\mu'$, $T'$ denote similar quantities for another planet: then, by Art. 177,

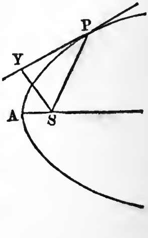
$$T=\frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}},\ T'=\frac{2\pi a'^{\frac{3}{2}}}{\sqrt{\mu'}}; \quad \text{therefore } \frac{T^2}{T'^2}=\frac{a^3}{a'^3}\frac{\mu'}{\mu}.$$

But by Kepler's third law $\frac{T^2}{T'^2}=\frac{a^3}{a'^3}$; therefore $\frac{\mu'}{\mu}=1$, so that $\mu'=\mu$. This shews that the constant which denotes the absolute force is the *same for all the planets;* so that the acceleration produced by the sun depends solely on the distance from the sun, and not on the nature of any particular planet.

179. By investigations similar to those in Arts. 174 and 175 it may be shewn that *if a body describes an hyperbola under the action of a force in a focus, the force varies inversely as the square of the distance.* If the body describes the branch which is the *nearer* to the focus, the force is attractive as in the case of the ellipse. But if the body describes the branch which is the more *remote* from the focus the force is *repulsive;* the body at any instant instead of moving along the tangent as it would if there were no central force, or deviating from the tangent on the side *towards* the centre of force as it would do if the force were attractive, deviates from the tangent on the side *remote* from the centre of force.

We proceed to consider the case of motion in a parabola round a force in the focus.

180. *If a body describes a parabola under the action of a force in the focus, the velocity at any point can be resolved into two equal constant velocities, one perpendicular to the axis of the parabola, and the other at right angles to the radius drawn from the body to the focus.*




Let $S$ be the focus, $P$ any point on the parabola, $A$ the vertex, $SY$ the perpendicular from $S$ on the tangent at $P$.

Let $AS=a$; and let $h$ denote twice the area described by the radius $SP$ in a unit of time.

$SY$ bisects the angle $ASP$; therefore the resultant of two velocities, each equal to $\frac{h}{2a}$, one along $SA$, and the other along

$SP$, is $\frac{2h}{2a} \cdot \frac{SY}{SP}$, that is, $\frac{h}{a} \cdot \frac{SY^2}{SY \cdot SP}$, that is, $\frac{h}{a} \cdot \frac{a \cdot SP}{SY \cdot SP}$, by the nature of the curve, that is, $\frac{h}{SY}$. But $\frac{h}{SY}$ is the magnitude of the velocity at $P$, and its direction is at right angles to $SY$. Hence the velocity at $P$ can be resolved into two velocities, each equal to $\frac{h}{2a}$, one perpendicular to $AS$, and the other at right angles to $SP$.

Hence it may be shewn, as in Art. 175, that *if a body describes a parabola under the action of a force in the focus, the force varies inversely as the square of the distance.* And if $\mu$ denote the absolute force, we have $\mu = \frac{h^2}{2a}$.

181. In the figure of Art. 174 we have the velocity at $P = \frac{h}{SY}$. Now by a property of the ellipse we might express $SY$ in terms of $SP$ and the major axis of the ellipse; and thus obtain another formula for the velocity at $P$. But instead of appealing to a property of the ellipse we can arrive at the result by the aid of mechanical principles, as we will now shew.

Let $v$ denote the whole velocity of the body at $P$; and let $v_1$ and $v_2$ have the same meanings as in Art. 175. Let $\alpha$ denote the angle $SPY$, and $\beta$ the angle between $YZ$ and $AH$ produced.

Suppose we resolve $v_1$ and $v_2$ along the tangent at $P$, and at right angles to it; then the algebraical sum of the former two components must be $v$, and the algebraical sum of the latter two components must be zero: that is,

$$v_1 \sin \alpha - v_2 \sin \beta = v, \qquad v_1 \cos \alpha - v_2 \cos \beta = 0.$$

From the second equation we have $\cos \beta = \frac{v_1 \cos \alpha}{v_2}$; substitute in the first equation; thus $v = v_1 \sin \alpha - \sqrt{(v_2^2 - v_1^2 \cos^2 \alpha)}$ $= v_1 \sin \alpha - \sqrt{(v_2^2 - v_1^2 + v_1^2 \sin^2 \alpha)}$.

Now

$$\sin\alpha = \frac{SY}{SP}; \text{ therefore } v_2^2 - v_1^2 + \frac{v_1^2 SY^2}{SP^2} = \left(\frac{v_1 SY}{SP} - \frac{h}{SY}\right)^2.$$

Hence
$$\frac{h^2}{SY^2} = v_2^2 - v_1^2 + \frac{2hv_1}{SP}.$$

Using the values of $v_1$ and $v_2$ which were found in Art. 174 we obtain

$$\frac{h^2}{SY^2} = \frac{2hv_1}{SP} - \frac{h^2a^2(1-e^2)}{b^4} = \frac{2hv_1}{SP} - \frac{h^2}{b^2}$$

$$= \frac{2\mu}{SP} - \frac{\mu}{a}, \text{ by Art. 175,}$$

that is,
$$v^2 = \frac{2\mu}{SP} - \frac{\mu}{a}.$$

In the same way we shall find that when a body moves in an *hyperbola* the square of the velocity $= \frac{2\mu}{SP} + \frac{\mu}{a}$.

## Examples. XV.

1. If a planet revolved round the sun in an orbit with a major axis four times that of the earth's orbit, determine the periodic time of the planet.

2. If a satellite revolved round the earth close to its surface, determine the periodic time of the satellite.

3. A body describes an ellipse under the action of a force in a focus: compare the velocity when it is nearest the focus with its velocity when it is furthest from the focus.

4. A body describes an ellipse under the action of a force to the focus $S$; if $H$ be the other focus shew that the velocity at any point $P$ may be resolved into two velocities, respectively at right angles to $SP$ and $HP$, and each varying as $HP$.

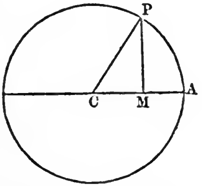
5. In the diagram of Art. 174 if $ASP = \theta$ shew that the velocity of the revolving body at $P$ may be resolved into $\frac{\mu}{h}(e + \cos\theta)$ perpendicular to $AC$ and $\frac{\mu\sin\theta}{h}$ parallel to $AC$.

## XVI. *Motion in an ellipse round the centre.*

182. We shall give in this Chapter investigations respecting the motion of a body in an ellipse round the centre. The results have not the practical interest which those in the preceding Chapter derive from their application to Astronomy; but the investigations will furnish valuable illustrations of mechanical principles.

183. It will be necessary to return to a result already established.

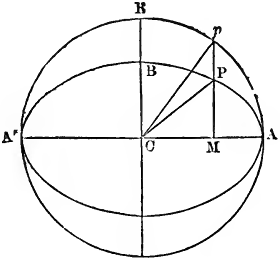
Suppose that a body describes a circle of radius $r$ with uniform velocity $v$. We have shewn that the body is acted on by a force to the centre of the circle of which the accelerating effect is $\frac{v^2}{r}$. Let $P$ be a point on the circumference of the circle, $C$ the centre, $CA$ a radius. Draw




$PM$ perpendicular to $CA$. Then a force of constant magnitude, acting along $PC$, may be represented by $PC$; and so may be resolved into two represented by $PM$ and $MC$ respectively. Thus we may say that if a body describes a circle with uniform velocity the forces acting on it may be represented by $PM$ and $MC$ respectively.

184. *A body describes an ellipse round a force in the centre: required to find the law of force.*

Let $ACA'$ be the major axis of the ellipse, $P$ any point on the ellipse; draw $PM$ perpendicular to the major axis. Produce $MP$ to meet the circle described on $AA'$ as diameter at $p$.




Now by Art. 156 the elliptic area $ACP$ varies as the time of moving from $A$ to $P$; and by a property of the ellipse the circular area $ACp$ bears a constant ratio to the elliptic area $ACP$. Hence, as $P$ moves, a body always occupying the position $p$ would describe the circle uniformly, and would therefore be acted on by a constant force along $pC$; or by Art. 183 it would be acted on by forces which we may represent by $pM$ and $MC$. Now $PM$ bears to $pM$ a constant ratio, by a property of the ellipse, so that the velocity of $P$ estimated parallel to $CB$ always bears a constant ratio to that of $p$ estimated in the same direction. Therefore the force parallel to $CB$ on $P$ bears to the force on $p$ in the same direction a constant ratio, namely that of $PM$ to $pM$; so that the force on $P$ in this direction may be represented by $PM$.

The force on $P$ parallel to $CA$ is the same as that on $p$, and so may be represented by $MC$.

Hence $P$ is acted on by two forces which may be denoted by $PM$ and $MC$ respectively; and the resultant of these will be a single force which may be denoted in magnitude and direction by $PC$.

Therefore the force required varies as the distance. Since the force varies as the distance $CP$ we may denote it by $\mu CP$, where $\mu$ is a constant; $\mu$ is usually called the *absolute force.*

185. Let $h$ denote twice the area described by $CP$ in a unit of time; and let $a$ and $b$ be the semiaxes of the ellipse: then will $h^2=\mu a^2b^2$. For the force at $B$ is $\mu b$, and therefore the force at $E$ is $\mu b \times \frac{a}{b}$, that is $\mu a$. Now the velocity of $P$ when at $B$ is the same as that of $p$ when at $E$; denote it by $v$: then, by Art. 163, $\frac{v^2}{a}=\mu a$, so that $v^2=\mu a^2$. But, by Art. 158, $v=\frac{h}{p}=\frac{h}{b}$; therefore $\frac{h^2}{b^2}=\mu a^2$, so that $h^2=\mu a^2b^2$.

186. *A body describes an ellipse under the action of a force in the centre: required to determine the periodic time.*

Let $a$ and $b$ denote the semiaxes of the ellipse; and $h$ twice the area described by the radius in a unit of time. By Art. 156 the periodic time

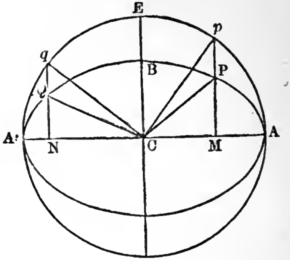
$$=\frac{\text{twice the area of the ellipse}}{h}=\frac{2\pi ab}{h}.$$

But $h=ab\sqrt{\mu}$, by Art. 185; therefore the periodic time

$$=\frac{2\pi}{\sqrt{\mu}}.$$

187. It will be seen that the result in the preceding Article is independent of the size of the ellipse; it will therefore hold even if we suppose $b$ indefinitely small, that is it will hold when the body moves in a straight line, oscillating backwards and forwards through the centre of force.

188. It is easy to give a geometrical representation of the velocity of a body moving in an ellipse under the action of a force in the centre.




Draw $Cq$ at right angles to $Cp$, meeting the circle at $q$; and draw $qN$ perpendicular to $AA'$, cutting the ellipse at $Q$.

The velocity of $p$ is constant in magnitude, and its direction is at right angles to $Cp$, so that it may be represented by $Cq$. The velocity represented by $Cq$ may be resolved into two represented by $CN$ and $Nq$ respectively. The velocity of $P$ parallel to $CA$ is equal to that of $p$, and the velocity of $P$ parallel to $CB$ is to that of $p$ as $b$ is to $a$: see Art. 184. Hence the velocity of $P$ parallel to $CA$ may be represented by $CN$, and that parallel to $CB$ by $NQ$, for $NQ = \frac{b}{a} Nq$. Thus the velocity of $P$ may be resolved into two components, denoted by $CN$ and $NQ$ respectively; so that the resultant velocity may be represented by $CQ$: that is, the resultant velocity of $P$ is parallel to $CQ$ in direction, and is proportional to $CQ$ in magnitude.

Since the velocity is proportional to $CQ$ in magnitude it will be equal to the product of $CQ$ into *some constant;* and by Art. 185, we see that this constant is $\sqrt{\mu}$.

## Examples. XVI.

1. A body describes an ellipse under the action of a force in the centre: if the greatest velocity is three times the least find the excentricity of the ellipse.

2. A body describes an ellipse under the action of a force in the centre: if the major axis is 20 feet and the greatest velocity 20 feet per second, find the periodic time.

## XVII. *Work.*

189. In many modern treatises on practical mechanics the term *work* is employed in a peculiar sense; and various useful facts and rules are conveniently stated by the aid of the term in this sense. We propose accordingly to give some explanations and illustrations which will enable the student to understand and apply such facts and rules.

190. The labour of men and animals, and the power furnished by nature in wind, water, and steam, are employed in performing operations of various kinds, such as drawing loads, raising weights, pumping water, sawing wood, and driving nails. In these and similar operations we may perceive one common quality which has been adopted as characteristic of work, and suggests the following definition: *work is the production of motion against resistance.*

191. This definition will not be fully appreciated at once; the beginner may be inclined to think that it will scarcely include every thing to which the term *work* is popularly applied: he will however find as he proceeds that the definition is wide enough for practical purposes.

According to this definition a man who merely supports a load without moving it does not work; for here there is resistance without motion. Also while a free body moves uniformly no work is performed; for here there is motion without resistance.

192. Work, like every other measurable thing, is measured by a unit of its own kind which we may choose at pleasure. The unit of work adopted in England is the work which is sufficient to overcome the resistance of a force of one pound through the space of one foot: or we may say practically that *the unit of work is the work done in raising a pound weight vertically through one foot.*

193. The term *foot-pound* is used in some books instead of *unit of work;* so that *foot-pound* may be considered as an abbreviation for *one pound weight raised vertically through one foot.*

194. The term *Horse-Power* is used in measuring the performances of steam engines. Boulton and Watt estimated that a horse could raise 33000 lbs. vertically through one foot in one minute; this estimate is probably too high, on the average, but it is still retained: so that a Horse-Power means a power which can perform 33000 units of work in a minute.

195. The term *duty* is also used with respect to steam engines; it means the quantity of work which is obtained by burning a given quantity of fuel. In good ordinary engines the duty of one pound weight of coal varies between 200000 and 500000.

196. Observations have been made of the amount of work which can be performed by men and by animals labouring in various ways; and the results are given in treatises on practical mechanics. The following table is an example: the first column states the kind of labour, the second column the number of hours in a day's labour, the third column the number of units of work performed in a minute.

| | | |
|---|---|---|
| Man raising his own weight on a ladder | 8 | 4230 |
| Man raising a weight with a cord and a Pully | 6 | 1560 |
| Man turning a windlass | 8 | 2600 |
| Man lifting earth with spade to the height of five feet | 10 | 470 |

197. Many examples can be given which involve nothing more than the application of the rules of Arithmetic to the measurement of work. We proceed now to some new propositions which arise from the combination of the definition of work with the known principles of mechanics.

198. *When weights are raised vertically through various heights the whole work is the same as that of raising a weight equal to the sum of the weights vertically from the first position of the centre of gravity of the system to the last.*

Suppose for example that there are three weights. Let $P$, $Q$, $R$ denote the weights; $p$, $q$, $r$ their respective heights above a fixed horizontal plane in the first position of the

system. Then by Arts. 119 and 146 of the Statics the distance of the centre of gravity of the system above the same fixed horizontal plane is

$$\frac{Pp+Qq+Rr}{P+Q+R}.$$

Now suppose the weights raised vertically through the heights $a$, $b$, $c$ respectively. Then the distance of the centre of gravity of the system in the new position above the same fixed horizontal plane is

$$\frac{P(p+a)+Q(q+b)+R(r+c)}{P+Q+R}.$$

Thus the vertical distance between the two positions of the centre of gravity of the system is

$$\frac{Pa+Qb+Rc}{P+Q+R}.$$

Now the work of raising a weight equal to the sum of the weights vertically through this space is the product of this space into the sum of the weights, that is into $P+Q+R$; hence this work is equal to $Pa+Qb+Rc$, that is to the sum of the work of raising $P$, $Q$, $R$ vertically through the heights $a$, $b$, $c$ respectively. In the same manner the proposition may be demonstrated whatever may be the number of the weights.

199. *The work done in raising a heavy body along a smooth inclined plane is equal to the work done in raising the same body through the corresponding vertical space.*

Let $\alpha$ be the inclination of the plane to the horizon, $W$ the weight moved, $s$ the distance along the plane through which the weight is moved. The weight $W$ may be resolved into two components, $W\cos\alpha$ at right angles to the plane and $W\sin\alpha$ down the plane; the latter is the component which resists motion along the plane. Hence the work done $= W\sin\alpha \times s = W \times s\sin\alpha$; and $s\sin\alpha$ is the vertical space corresponding to the space $s$ on the plane. This establishes the proposition.

200. *If a body of weight* W *be dragged along a rough horizontal Plane through a space* s, *and* $\mu$ *be the coefficient of friction for motion, the work done is* $\mu$Ws.

For the weight being $W$ the force which resists the horizontal motion is $\mu W$; and therefore the work done is $\mu Ws$.

201. *If a body of weight* W *be dragged up a rough Plane inclined to the horizon at an angle* $\alpha$ *through a space* s, *and* $\mu$ *be the coefficient of friction for motion, the work done is* $W(\sin\alpha + \mu\cos\alpha)s$.

For the weight $W$ may be resolved into two components, $W\cos\alpha$ at right angles to the Plane and $W\sin\alpha$ along the Plane. The work done consists of two parts, namely, raising the weight along the Plane, and overcoming the resistance along the Plane; the former part is $W\sin\alpha \times s$, and the latter part is $\mu W\cos\alpha \times s$. Hence the whole work is $W(\sin\alpha + \mu\cos\alpha)s$.

This may also be obtained in another way. By Art. 86 the resolved force down the Plane is $W(\sin\alpha + \mu\cos\alpha)$, and therefore by the definition of work $W(\sin\alpha + \mu\cos\alpha)s$ is the work done in dragging the body up the Plane.

Since $s\sin\alpha$ represents the *vertical* height through which the weight is raised, and $s\cos\alpha$ the *horizontal* space through which it is drawn, we may say that the work consists of two parts, one being that which would be required to raise the weight through the vertical height passed over, and the other that which would be required to overcome the friction supposing the Plane to be horizontal. In most cases which occur in practice $\alpha$ is so small that $\cos\alpha$ may without any important error be taken as equal to unity, and the expression for the work becomes $Ws\sin\alpha + \mu Ws$.

Similarly if a body be dragged through a space $s$ *down* an Inclined Plane which is too rough for the body to slide down by itself, the work done is $W(\mu\cos\alpha - \sin\alpha)s$.

202. The preceding Article may be applied to the case of carriages drawn along a common road or a railroad; in this there is indeed a rotatory motion of the wheels which

is not contemplated in the preceding Article; but the weight of the wheels will in general be small compared with the weight of the whole mass which is moved, and we will assume that no important error will arise from neglecting the rotatory motion.

The numerical value of $\mu$ will depend on various circumstances. Take the case of a cart on a common road; then observation indicates that the value of $\mu$ depends on the size of the wheels and on the velocity of motion as well as on the nature of the road. For a cart having wheels four feet in diameter drawn with a velocity of six miles an hour along a good road, the value of $\mu$ may lie between $\frac{1}{30}$ and $\frac{1}{40}$. Again, consider a train drawn along a railroad; then observation indicates that the value of $\mu$ depends on the velocity. For a velocity of 30 miles an hour the value will be about $\frac{16}{2240}$, that is the friction is about 16 lbs. per ton, estimated on the whole weight of the engine and the load. There is however besides this the resistance of the air, which depends on the square of the velocity and the area of the frontage of the train.

203. There is one mode in which the labour of men and of animals is employed which is not directly comparable with the application of a force to raise a weight, namely, that of carrying burdens along a horizontal road. It seems that a portion of the labour is spent in merely supporting the burden, and this portion does no *work* in the sense in which the term is used here: the remainder of the labour does the work of carrying the burden. By observation we can find the amount of useful effect which can be produced by this mode of labour; thus, for example, it is said that a porter walking with a burden on his back through a day of seven hours long can carry a weight of 90 lbs. through 145 feet in a minute. Here the useful effect is equivalent to 13050 units of work per minute. But this must not be taken to measure the labour of the porter; for, as we have said, some labour is spent in merely supporting the bur-

den: moreover some labour is also spent by the porter in carrying the weight of his own body.

See Young's *Lectures on Natural Philosophy*, Lecture XII., and Whewell's *Mechanics of Engineering*, page 178.

204. In the Statics we investigated the conditions of equilibrium of the simple machines and of some compound machines. In practice however machines are generally used not to maintain equilibrium but to assist in doing work. By the aid of machines the labour of men and of animals and the powers furnished by nature are transmitted and applied in various ways. Now it is a general principle that if we set aside friction and the weights of the parts of a machine, then *the work applied to the machine is equal to the work done by the machine:* this principle is called by some writers the *principle of work*.

205. It would be impossible in an elementary treatise to *demonstrate* the principle just stated, or even fully to explain its meaning: we will however give two simple illustrations.

Take the case of the Wheel and Axle; see Art. 236 of the Statics. Suppose $P$ moving *uniformly* downwards, and therefore $W$ moving *uniformly* upwards. Then the relation of $P$ to $W$ is found to be the same as in the state of equilibrium. Hence it follows that the product of $P$ into the vertical space which it describes is equal to the product of $W$ into the vertical space which it describes; that is in this case the work applied to the machine is equal to the work done by the machine.

Again, suppose a body of weight $W$ drawn up an Inclined Plane by means of a string which passes over a smooth Pully fixed at the top of the Plane and has a weight $P$ attached to it which hangs vertically.

Let $a$ be the inclination of the Plane. It may be shewn that if the motion is *uniform* we shall have the same relation between $P$ and $W$ as in the state of equilibrium: see Statics, Arts. 211, 244, and Dynamics, Art. 92. Hence it will follow that the product of $P$ into the vertical space which it describes is equal to the product of $W$ into the

space which it describes *resolved vertically*. Thus the work applied to the machine is equal to the work done by the machine.

206. But in practice, owing to friction and the weights of the parts of a machine, the principle of Art. 204 must be modified. The whole work performed by a machine is distinguished into two parts, namely, the *useful* part and the *lost* part: the useful work is that which the machine is designed to produce; the lost work is that which is not wanted but which is unavoidably produced, such for example as the wearing away by friction of the machine itself. It is still true that the work applied to a machine is equal to the whole work, useful and lost, done by the machine; and consequently the *useful* work done by the machine is always less than the work applied to the machine.

207. The ratio of the useful work done by a machine to the work applied to the machine is called the *efficiency* of the machine, or sometimes the *modulus* of the machine. The efficiency is thus a fraction, and it is of course the object of inventors and improvers to bring this fraction as near to unity as possible.

208. *Accumulated Work.* If a body is moving it is said to have work accumulated in it. In fact if a body possesses any velocity it can be made to do work by parting with that velocity. For example, a cannon ball in motion can penetrate a resisting body; water flowing against a water-wheel will turn the wheel.

We will now shew how the amount of work accumulated in a body may be conveniently estimated.

209. Let a body of mass $M$ be moving with a velocity $v$; let a constant force $F$ acting on the body through a space $s$ bring it to rest; then we shall take $Fs$ as the measure of the work accumulated in the body.

We know by Art. 87 that the acceleration is $\frac{F}{M}$; therefore $v^2=\frac{2F}{M}s$, by Art. 42: thus $Fs=\frac{Mv^2}{2}$. Hence we may say that the work accumulated in a moving body is

measured by *half the Vis Viva of the body:* see Arts. 85 and 107.

Let $h$ be the height through which the body must fall to acquire the velocity $v$, and $W$ the weight of the body. Then $v^2=2gh$; and $W=Mg$: therefore $Wh=Fs$. Hence we may say that the work accumulated in a moving body is measured *by the product of the weight of the body into the height through which it must fall to acquire the velocity.*

We know from Art. 124 that by the collision of bodies there is always a loss of *Vis Viva* if the elasticity be imperfect; so we may say in such a case that there is always a loss of accumulated work.

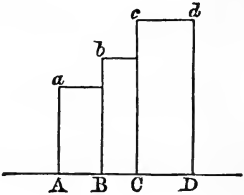
210. Suppose a body of weight $W$ to be moving in a straight line, and urged on by a force in that straight line; if in moving through a space $s$ the velocity changes from $u$ to $v$ the work done on the body as it moves through that space is $\frac{W}{2g}(v^2-u^2)$: if there be one force urging the body on, and another force resisting the body, this expression gives the excess of the work done by the former force over the work done by the latter force.

211. We have seen that a body in motion may be said to have work accumulated in it, because it may be made to do work by parting with its velocity; in like manner, if a body be in such a position that it can fall and thus acquire velocity, it may be considered to possess a store of work ready to be used. Thus suppose that a weight has been drawn up to a certain height; it may then be allowed to fall, and do work in various ways, as for instance in driving piles. So men may employ their labour in ascending to a high point, and then descending as weights in a machine. Water which can fall from a certain height contains a store of work the amount of which is measured by the product of the weight of the water into the vertical descent: this store of work can be used to turn a water-wheel.

212. Hitherto we have supposed that the force which performs work is a *constant* force; we will now make some remarks concerning the work performed by a *variable*

force, that is a force which is not constant: but the complete discussion of this subject is beyond the range of an elementary treatise.

213. Suppose we have a force of 4 lbs., which acts through 2 feet; then let the force be changed into a force of 5 lbs., and act through 2 feet, and then be changed into a force of 7 lbs., and act through 3 feet. Then the whole work done is



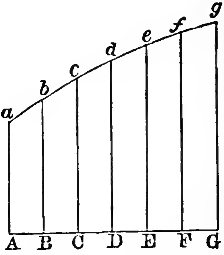

$4 \times 2 + 5 \times 2 + 7 \times 3$, that is 39.

Now there is a convenient mode of representing this calculation. Take on a straight line successive lengths to represent the spaces through which the forces act: thus let $AB$ represent two feet, $BC$ two feet, and $CD$ three feet. Draw at $A$, $B$, and $C$ straight lines at right angles to $ABC$ to represent the forces which act respectively through the distances represented by $AB$, $BC$, and $CD$: thus let $Aa$ represent the force of 4 lbs., $Bb$ the force of 5 lbs., and $Cc$ the force of 7 lbs. Complete the rectangles $Ba$, $Cb$, and $Dc$; *then the area of each rectangle represents the work done by the corresponding force.* This is obvious, because the area of a rectangle is equal to the product of its base into its altitude.

*Hence the sum of all the areas represents the whole work.*

214. Now let us suppose that the changes in the value of the force are *more gradual* than in the preceding Example; that is, suppose the changes to be more frequent, but each change to be of less amount. For instance, suppose a force of 5 lbs. to act through 6 inches, then a force of $5\frac{1}{3}$ lbs. to act through 10 inches, then a force of $5\frac{2}{3}$ lbs. to act through 8 inches, then let the force change to 6 lbs.; and so on. Still the above geometrical mode of representing the calculation may be conveniently applied. Instead of the rectangle $Ba$ we should now have three rectangles, the breadths representing respectively $\frac{1}{2}$, $\frac{5}{6}$, and $\frac{2}{3}$ of a foot, and the heights 5, $5\frac{1}{3}$, and $5\frac{2}{3}$ lbs.

215. By proceeding in this way we can obtain the conception of a force which is always changing its value, so that it does not remain constant even for a very small space. Let a straight line $AG$ represent the whole space through which a variable force acts; conceive that at every point of $AG$ straight lines are drawn at right angles to $AG$, so that any straight line $Dd$ represents the amount of the force at the point $D$. Then the other extremities of these straight lines will form a curve $adg$: *and the whole area bounded by the curve and the straight lines will represent the whole work done.*




216. No exact Rule can be given for finding the area of such a figure as that of Art. 215, which will apply whatever may be the form of the curve: but a Rule called *Simpson's Rule* will furnish results which are sufficient approximations to the truth for many practical purposes. The straight lines $Aa$, $Bb$,...$Gg$ are called *ordinates;* the values of an odd number of them, drawn at equal distances, are supposed to be known; then the Rule for finding the area is this: *Add together the first ordinate, the last ordinate, twice the sum of all the other odd ordinates, and four times the sum of all the even ordinates; multiply the result by one-third of the common distance between two adjacent ordinates.*

See *Mensuration for Beginners*, Chapter XVIII.

217. An important application of the preceding method occurs in the *Steam Indicator.* While the piston of a steam engine is making a stroke the pressure of the steam on the piston varies. By a suitable contrivance the amount of the pressure in any position is registered: in fact a curve is drawn corresponding to the curve $adg$ in Art. 215. Then by the Rule of Art. 216 the whole work is calculated.

218. We may remark that the process of Art. 215 is applicable to other investigations as well as to that of the work done by a variable force.

For example, consider the velocity generated in a given time in a particle by a *variable* force. Here let the straight line $AG$ represent the whole time during which the force acts; and let the straight lines at right angles to this represent the force at the corresponding instants. Then the area will represent the whole velocity generated in the given time.

Again, consider the space described in a given time by a particle moving with *variable* velocity. Here let the straight line $AG$ represent the whole time of motion; and let the straight lines at right angles to this represent the velocity at the corresponding instants. Then the area will represent the whole space described in the given time.

## Examples. XVII.

1. Find how many units of work are performed in raising 2 cwt. of coal from a pit 50 fathoms deep.

2. Find how many cubic feet of water an engine of 40 Horse-Power will raise in an hour from a mine 80 fathoms deep, supposing a cubic foot of water to weigh 1000 ounces.

3. Find how many bricks a labourer could raise in a day of 6 hours to the height of 20 feet by the aid of a cord and a Pully, supposing a brick to weigh 8 lbs. See Art. 196.

4. Find the Horse-Power of an engine which is to move at the rate of 30 miles an hour, the weight of the engine and load being 50 tons, and the resistance from friction 16 lbs. per ton.

5. Find the Horse-Power of an engine which is to move at the rate of 20 miles per hour up an incline which rises 1 foot in 100, the weight of the engine and load being 60 tons, and the resistance from friction 12 lbs. per ton.

6. A well is to be made 20 feet deep, and 4 feet in diameter: find the work in raising the material, supposing that a cubic foot of it weighs 140 lbs.

7. Supposing that a man by turning a windlass can perform 2600 units of work per minute, find how many cubic feet of water he can raise to the height of 24 feet in 8 hours; the efficiency of the machine being ·6.

8. If an engine of 50 Horse-Power raise 2860 cubic feet of water per hour from a mine 60 fathoms deep, find the efficiency of the engine.

9. Find the work accumulated in a body which weighs 300 lbs. and has a velocity of 64 feet per second.

10. Find the useful Horse-Power of a water-wheel, supposing the stream to be 5 feet broad and 2 feet deep and to flow with a velocity of 30 feet per minute; the height of the fall being 14 feet, and the efficiency of the machine ·65.

11. Determine the Horse-Power of a steam engine which will raise 30 cubic feet of water per minute from a mine 440 feet deep.

12. It is found that a man with a capstan can do 3120 units of work per minute for 8 hours a day, but if he carries weights up a ladder he can do only 1120 units of work per minute: find in each case how long he will take in raising 36 tons from a depth of 130 feet.

13. A steam engine is required to raise 70 cubic feet of water per minute from a depth of 800 feet: find how many tons of coal will be required per day of 24 hours, supposing the duty of the engine to be 250000 for a lb. of coal.

14. A variable force has acted through 3 feet; the value of the force taken at seven successive equidistant points, including the first and the last, is in lbs. 189, 151·2, 126, 108, 94·5, 84, 75·6: find the whole work done.

15. A railway truck weighs $m$ tons; a horse draws it along horizontally, the resistance being $n$ lbs. per ton; in passing over a space $s$ the velocity changes from $u$ to $v$: find the work done by the horse in this space.

16. A stream is $a$ feet broad, and $b$ feet deep, and flows at the rate of $c$ feet per hour; there is a fall of $h$ feet; the water turns a machine of which the efficiency is $e$: find the number of bushels of corn which the machine can grind in an hour, supposing that it requires $m$ units of work per minute for one hour to grind a bushel.

17. A shaft $a$ feet in depth is full of water: find the depth of the surface of the water when one quarter of the work required to empty the shaft has been done.

18. Find at what rate an engine of 30 Horse-Power could draw a train weighing 50 tons up an incline of 1 in 280, the resistance from friction being 7 lbs. per ton.

19. The French unit of work is the work done in raising a kilogramme vertically through a metre: find how many English units of work are equivalent to a French unit, taking the metre at 39·37 inches, and the kilogramme at 15432 grains.

20. According to Navier it requires 43333 French units of work to saw through a square metre of green oak: find how many English units of work are required to saw through a square foot of green oak.

21. A saw-mill was supplied with 111500 units of work per minute: in eleven minutes 13 square feet of green oak were sawn by the mill: find the ratio of the lost work to the useful work. See the preceding Example.

22. Find the useful work done by a fire-engine per second which discharges every second 13 lbs. of water with a velocity of 50 feet per second.

23. Find how many units of work are stored up in a mill-pond which is 100 feet long, 50 feet broad, and 3 feet deep, and has a fall of 8 feet.

24. In pile-driving 38 men raised a rammer 12 times in an hour; the weight of the rammer was 12 cwt., and the height through which it was raised 140 feet: find the work done by one man in a minute.

## XVIII. *D'Alembert's Principle.*

219. The three Laws of Motion which have been already given are found to be sufficient for the discussion of all problems connected with the motion of *particles.* The mathematical investigations may be in some cases extremely long and difficult, but no new mechanical principles are required. In the discussion of problems connected with the motion of *bodies* however another Law is required; this was first explicitly stated by D'Alembert, and is called *D'Alembert's Principle:* we shall now give some account of it.

220. The particles of a solid body are acted on by various forces which bind the particles together, and which are usually called *molecular forces.* If we knew the direction and the amount of the molecular forces which act on an assigned particle as well as the other forces, like gravity, which act on it, we could determine the motion of the particle. But the nature of the molecular forces is unknown, and it is the object of D'Alembert's Principle to obtain results respecting the motion of a body without considering the molecular forces which act on the particles of the body.

221. It is necessary to explain the distinction between *impressed force* and *effective force.* By the *impressed* force on a particle is meant the external force which acts on it, that is all the force which acts on it *except* the molecular force. By the *effective* force is meant the force which is just sufficient to produce the actual motion of the particle. Thus the effective force is equivalent to the resultant of the impressed force and the molecular force. By *reversed effective force* is meant a force equal in magnitude and opposite in direction to the effective force. These definitions have to be given before we state D'Alembert's

Principle, but they will probably not be fully understood until the Principle and some of its applications have been considered.

222. D'Alembert's Principle may be stated thus: *the impressed forces together with the reversed effective forces in a body or system of bodies satisfy the conditions of equilibrium for the body or system of bodies.*

223. It must be observed that the utility of the Principle depends on the fact that we can find expressions for the effective forces in terms of the motion produced.

224. We will give some illustrations of the Principle from cases already considered: these illustrations will serve to explain the meaning of the terms which are used, although they will supply little notion of the importance and the extent of the Principle.

225. Suppose a particle of mass $m$ to describe a circle of radius $r$ with uniform velocity $v$; then, as we have shewn in Art. 166, the force acting on the particle must be equal to $\frac{mv^2}{r}$, and must tend towards the centre of the circle. Thus if $F$ denote the force, $F - \frac{mv^2}{r} = 0$. Now this equation may be considered an example of the Principle; $F$ is the impressed force, and $-\frac{mv^2}{r}$ is the reversed effective force: thus the equation expresses the condition for the equilibrium of the impressed force and the reversed effective force.

226. If a particle is moving in a straight line with uniform velocity the effective force is zero; for no force is required to maintain such a motion. Suppose a particle of mass $m$ to be moving in a straight line, the velocity not being uniform; let $f$ denote the acceleration at any instant, that is the velocity which is added in a unit of time if the force be uniform, or which would be added in a unit of time if the force were to continue uniform during that unit: them $mf$ is the effective force at that instant.

227. We shall now investigate the value of the effective force when a particle describes a circle, the velocity not being uniform. Take the diagram of Art. 163; let $v$ denote the velocity at $P$, and $v_1$ the velocity at $Q$. Suppose that the force acting at $P$ is resolved into two components, one in the direction $PS$, and the other in the direction $PT$. Let the angle $PSQ$ be denoted by $2\phi$. At $Q$ the velocity is along $TQ$ produced; and it may be resolved into $v_1 \cos 2\phi$ along a straight line parallel to $PT$, and $v_1 \sin 2\phi$ along a straight line parallel to $PS$. Thus in passing from $P$ to $Q$ the change of velocity is $v_1 \cos 2\phi - v$ in the direction $PT$, and $v_1 \sin 2\phi$ in the direction $PS$. Let $t$ denote the time in which the particle moves from $P$ to $Q$. Then if we assume the force to continue uniform we have $\frac{v_1 \cos 2\phi - v}{t}$ as the measure of the force along $PT$; and this expression is equal to $\frac{v_1 - v}{t} - \frac{2v_1 \sin^2 \phi}{t}$. Now, as in Art. 163, we have $2r\phi = v_1 t$ when $t$ is made small enough, so that the above expression is equal to $\frac{v_1 - v}{t} - \frac{2{v_1}^2 \sin^2 \phi}{2r\phi}$. When $\phi$ is made indefinitely small the second term vanishes, and the first term remains alone: this is the same as if the motion at $P$ were along the tangent there instead of along the arc of the circle. Thus if $m$ be the mass of the particle, and $f$ the acceleration at $P$ considering the motion to be along the tangent $PT$, then the effective force in the direction $PT$ is $mf$. The effective force in the direction $PS$ is $\frac{mv^2}{r}$; this is obtained in the manner of Art. 163: the only difference is that the equation $2r\phi = vt$ which is used is now not absolutely true, but only true when $t$ is taken small enough.

228. We can now give some notion of the way in which the important result stated in Art. 135 is obtained. Suppose $m_1, m_2, m_3, \ldots$ to denote the masses of the particles of a rigid body; and $f_1, f_2, f_3, \ldots$ the respective accelerations of the particles estimated parallel to an assigned direction. Let $F$ denote the sum of the impressed forces acting

parallel to this direction; then by *Statics*, Art. 90, and D'Alembert's Principle

$$F - m_1 f_1 - m_2 f_2 - m_3 f_3 - \ldots = 0.$$

But if $f$ be the acceleration of the centre of gravity we have, by Art. 131,

$$(m_1 + m_2 + m_3 + \ldots) f = m_1 f_1 + m_2 f_2 + m_3 f_3 + \ldots$$

Hence $$F - (m_1 + m_2 + m_3 + \ldots) f = 0:$$

that is the motion of the centre of gravity parallel to the assigned direction is the same as that of a particle having a mass equal to the mass of the rigid body, and acted on by forces parallel to those which act on the rigid body. The Article 90 of the *Statics* to which we appeal contemplates forces in only *one* plane, though the result can be shewn to be true universally: but even with this restriction we obtain a good notion of the way in which Art. 135 is established.

229. Suppose that $F=0$ in the preceding Article, then $f=0$; but when $f=0$ the centre of gravity is either at rest, or moving uniformly in a straight line, so far as regards the assigned direction: hence if the forces which act on a body in an assigned direction vanish, the centre of gravity either has no motion in that direction or moves uniformly in it. Therefore if no forces act on a rigid body the centre of gravity either remains at rest or moves uniformly in some straight line. Thus it appears that by the aid of D'Alembert's Principle we can make a statement with regard to a *rigid body* analogous to that which is made in the First Law of Motion with respect to a *particle*.

230. With regard to the truth of D'Alembert's Principle remarks may be made similar to those which have been already applied to the Laws of Motion: see Arts. 12 and 140. But although the Principle may be confirmed in an indirect manner yet the evidence for it is probably somewhat short of that which establishes the truth of the Laws of Motion. Strictly speaking it is only by means of D'Alembert's Principle that we are justified in treating the heavenly bodies as particles in our theories; and thus it

may be said that Astronomy verifies the truth of D'Alembert's Principle as well as that of the Laws of Motion : but owing to the small dimensions of these bodies compared with the enormous distances which separate them, and to their nearly spherical forms, we may calculate the motions as if these bodies were particles, by means of special considerations of a reasonable character, without appealing to the Principle. There is indeed one very important problem in Astronomy, namely that of Precession and Nutation, which distinctly involves the Principle; but then various assumptions have to be made in order to arrive at numerical results, so that the testimony to the truth of the Principle is not decisive. If we regard the evidence of Astronomy as inconclusive we must turn to other cases in which theory involving D'Alembert's Principle may be compared with observation and experiment; probably the most satisfactory instances of this kind are those which depend on the motion of pendulums, which will be considered in Chapter XX.

231. Perhaps however on due reflection the Principle will appear to involve only what may be readily accepted. The impressed forces and the molecular forces, together with the reversed effective forces, are in equilibrium by definition; thus D'Alembert's Principle amounts to the assertion that the molecular forces will be in equilibrium taken alone. It is plain from observation that this is practically the case when the body is not in motion; thus we merely assume that the molecular forces when the body is in motion satisfy the same conditions as they do when the body is not in motion.

## EXAMPLES. XVIII.

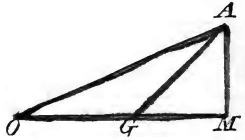
1. Point out the impressed forces and the effective force in the case of Art. 92.

2. Point out the impressed forces and the effective force in the case of Art. 168.

## XIX. *Moment of Inertia.*

232. Let the mass of every particle of a body be multiplied into the square of its distance from an assigned straight line; the sum of these products is called the *moment of inertia* of the body about that straight line. The straight line is often called an *axis*. In the discussion of problems respecting the motion of rigid bodies the moment of inertia occurs very frequently, so that it becomes necessary to consider this subject in detail: we shall demonstrate some general results, and shall calculate the value of the moment of inertia in various special cases.

233. *The moment of inertia of any body about an assigned axis is equal to the moment of inertia of the body about a parallel axis through the centre of gravity of the body, increased by the product of the mass of the body into the square of the distance between the axes.*




Let $m$ be the mass of one particle of the body; let this particle be at $A$. Suppose a plane through $A$, at right angles to the assigned axis, to meet that axis at $O$, and to meet the parallel axis through the centre of gravity at $G$. From $A$ draw a straight line $AM$, perpendicular to $OG$ or to $OG$ produced. Let $GM = x$, where $x$ is a positive or a negative quantity according as $M$ is to the right or left of $G$. By Euclid II. 12 and 13 we have $OA^2 = OG^2 + GA^2 + 2OG \, . \, x$; therefore

$$m \, . \, OA^2 = m \, . \, OG^2 + m \, . \, GA^2 + 2OG \, . \, m \, . \, x.$$

A similar result holds with respect to every particle of the body. Hence we see that the moment of inertia with respect to the assigned axis is composed of three parts: namely, first the sum of such terms as $m \, . \, OG^2$, and this will be equal to the product of the mass of the body into $OG^2$; secondly the sum of such terms as $m \, . \, GA^2$, and this will be the moment of inertia of the body about the axis through $G$; and thirdly the sum of such terms as $2OG \, . \, m \, . \, x$, which is zero by Art. 146 of the *Statics*. Hence the moment

of inertia about the assigned axis has the value stated in the proposition.

234. For the sake of abbreviation the symbol $\Sigma$ is used to indicate the sum of terms of any specified kind; thus $\Sigma m . OA^2$ means the sum of such terms as $m . OA^2$. The context will always make it evident what collection of terms we are considering. In future unless the restriction is expressly removed we shall confine ourselves to *plane figures;* that is the bodies we consider are to be supposed of uniform thickness, but that thickness is to be infinitesimal. Also the straight lines we have to use will be in the plane of the figure unless the contrary is stated.

235. *The moment of inertia of any plane figure about any straight line at right angles to its plane is equal to the sum of the moments of inertia about any two straight lines at right angles to each other in the plane which intersect at the first straight line.*

Let $O$ denote any point in the plane; through $O$ draw any two straight lines at right angles to each other in the plane. Suppose one particle of the body to be at the point $T$ of the plane; let $m$ be the mass of the particle. Denote by $x$ and $y$ respectively the perpendiculars from $T$ on the two straight lines in the plane. Then $x^2+y^2$ denotes the square of the distance of $T$ from the straight line drawn through $O$ at right angles to the plane; the moment of inertia of the body about this straight line then is $\Sigma m(x^2+y^2)$, that is $\Sigma mx^2 + \Sigma my^2$. But $\Sigma mx^2$ is the moment of inertia about one of the straight lines in the plane, and $\Sigma my^2$ is the moment of inertia about the other. Thus the proposition is established.

236. Hence when the moments of inertia of a plane figure about two straight lines at right angles to each other in the plane of the figure are known, the moment of inertia about the straight line at right angles to the plane through the common point is known immediately. The moment of inertia about any straight line *in the plane* through the common point is also connected with the moments of inertia about the first two straight lines, but

the relation is not so simple as in the case of the straight line at right angles to the plane. To this we now proceed.

237. Through any point $O$ in the plane of a plane figure, draw any two straight lines $OX$ and $OY$ at right angles to each other in the plane; also through $O$ draw any straight line $OL$ in the plane. It is required to connect the moment of inertia about $OL$ with the moments of inertia about $OX$ and $OY$.

Let $a$ denote the angle $LOX$. Suppose one particle of the body to be at the point $T$ of the plane; let $m$ be the mass of the particle. Let $OT$ be denoted by $r$; and let the perpendicular from $T$ on $OX$ be denoted by $y$, and the perpendicular from $T$ on $OY$ by $x$. Let $A$ denote the moment of inertia of the body round $OX$, and $B$ the moment of inertia round $OY$.

$$\begin{aligned}\text{The perpendicular from } T \text{ on } OL &= r \sin TOL \\ &= r \sin (TOX - LOX) = r \sin (TOX - a) \\ &= r \sin TOX \cos a - r \cos TOX \sin a \\ &= y \cos a - x \sin a.\end{aligned}$$

$$\begin{aligned}\text{The moment of inertia round } OL &= \Sigma m (y \cos a - x \sin a)^2 \\ &= \Sigma m (y^2 \cos^2 a + x^2 \sin^2 a - 2xy \sin a \cos a) \\ &= \cos^2 a \, \Sigma m y^2 + \sin^2 a \, \Sigma m x^2 - 2 \sin a \cos a \, \Sigma mxy \\ &= A \cos^2 a + B \sin^2 a - 2 \sin a \cos a \, \Sigma mxy.\end{aligned}$$

Thus we see that we cannot determine the moment of inertia about $OL$ from the three quantities $A$, $B$, and $a$; because the preceding expression involves another quantity, namely $\Sigma mxy$.

238. The quantities $x$ and $y$ of the preceding Article are called the *co-ordinates* of $T$ with respect to the axes $OX$ and $OY$; the point $O$ is called the *origin;* the quantity $\Sigma mxy$ is called the *product of inertia* of the body for the axes $OX$ and $OY$. If at a point in a plane figure two straight lines can be drawn at right angles to each other in the plane such that the corresponding product of inertia vanishes, the two straight lines are called *principal axes* of the plane figure at the point.

239. We shall now shew that at every point in the plane of a plane figure principal axes exist. The moment of inertia of an assigned body about an assigned axis is by definition a finite positive quantity. Thus among all the axes which pass through a given point there must be one for which the moment of inertia is greater than it is for all other axes, or at least for which it is as great as it is for any other axis. Let $O$ denote the point considered, and suppose that $OX$ is an axis for which the moment of inertia is as great as it is for any axis through $O$. Let $OY$ be at right angles to $OX$, and $OL$ any other straight line through $O$; and denote the angle $LOX$ by $\theta$. Let the moment of inertia about $OX$ be denoted by $A$, that about $OY$ by $B$, and that about $OL$ by $Q$. Also let $P$ denote the product of inertia with respect to $OX$ and $OY$. Then by Art. 237,

$$Q = A\cos^2\theta + B\sin^2\theta - 2P\sin\theta\cos\theta.$$

Now by hypothesis $A - Q$ cannot be negative; but

$$\begin{aligned} A - Q &= A - A\cos^2\theta - B\sin^2\theta + 2P\sin\theta\cos\theta \\ &= \sin^2\theta\,(A - B + 2P\cot\theta), \end{aligned}$$

so that the sign of this expression is the same as the sign of $A - B + 2P\cot\theta$. But if $P$ have any value different from zero we can make $2P\cot\theta$ numerically as great as we please by taking $\theta$ small enough: thus if $P$ were *positive*, by taking $\theta$ very small and negative we should make $A - B + 2P\cot\theta$ negative; and if $P$ were *negative* we should attain the same object by taking $\theta$ very small and positive. Hence it follows that $P$ must be zero, for otherwise $A - Q$ could be rendered negative; and since $P$ is zero, $OX$ and $OY$ constitute *principal axes.*

Thus $Q = A\cos^2\theta + B\sin^2\theta$; and as this can be put in the form $Q = B + (A - B)\cos^2\theta$ it follows that $Q$ cannot be less than $B$. Thus principal axes at a point have this property with respect to all axes through the point: no moment of inertia can be greater than that for one of these axes, and no moment of inertia can be less than that for the other axis. If $A = B$ we have $Q = A$; and then the moment of inertia is the same for all axes through the point.

240. We may express this result with sufficient clearness by saying that the axes of greatest and least moments of inertia for an assigned point are at right angles to each other; and we see that this is consistent with Art. 235. For that Article shews that the sum of the moments of inertia about two axes at right angles to each through an assigned point is *constant*, namely equal to the moment of inertia about the axis through the point at right angles to the plane: hence when one of the two has the greatest possible value the other must have the least possible value.

241. The position of the axes for which the product of inertia vanishes, that is the position of principal axes, is often sufficiently obvious. Suppose, for example, we require the position of principal axes for a rectangle when the given point is the centre. Draw through the centre straight lines parallel to the sides of the rectangle; then these will be the principal axes. For consider a particle of mass $m$, at a point of which the coordinates are $x$ and $y$; it is plain that there is also a corresponding particle of mass $m$ at the point of which the coordinates are $x$ and $-y$. Hence we see that $\Sigma mxy$ consists of terms which may be arranged in pairs, so that the two terms in a pair are numerically equal but of opposite signs; and therefore $\Sigma mxy$ vanishes.

242. The value of the product of inertia at any point may be made to depend on the value of the product of inertia for parallel axes through the centre of gravity. Let $x$ and $y$ be the coordinates of a particle of mass $m$ referred to axes through any assigned point; and let $x'$ and $y'$ be the coordinates of the same particle referred to parallel axes through the centre of gravity; also let $h$ and $k$ be the coordinates of the centre of gravity referred to the first pair of axes. Then

$$x = x' + h, \qquad y = y' + k;$$

therefore
$$\Sigma mxy = \Sigma m(x' + h)(y' + k)$$
$$= \Sigma mx'y' + h\Sigma my' + k\Sigma mx' + hk\Sigma m.$$

But by Art. 146 of the *Statics*

$$\Sigma my' = 0, \qquad \Sigma mx' = 0;$$

therefore
$$\Sigma mxy = \Sigma mx'y' + hk\Sigma m.$$

243. The result of the preceding Article will often enable us to calculate the value of the product of inertia for an assigned origin and axes. Suppose, for example, that we require the product of inertia in the case of a rectangle, when the origin is at a corner, and the axes are the edges which meet at that corner. Then by Art. 241 we have $\Sigma mx'y'=0$; and therefore $\Sigma mxy=hk\Sigma m$; and $h$ and $k$ are known, being half the lengths of the edges of the rectangle to which they are respectively parallel.

244. If the moments of inertia about three axes passing through a point are known, the position of the principal axes at that point and the moments of inertia about them can be determined. Let $Q$, $R$, $S$ denote the three known moments of inertia; suppose $\beta$ the angle between the axes corresponding to $Q$ and $R$, and $\gamma$ the angle between the axes corresponding to $Q$ and $S$. Let $\theta$ denote the unknown angle between one of the principal axes, and the axis corresponding to $Q$; let $A$ denote the moment of inertia about this principal axis, and $B$ the moment of inertia about the other principal axis. Then, by Art. 239,

$$Q=A\cos^2\theta+B\sin^2\theta,$$
$$R=A\cos^2(\theta+\beta)+B\sin^2(\theta+\beta),$$
$$S=A\cos^2(\theta+\gamma)+B\sin^2(\theta+\gamma).$$

These may be written

$$Q=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)\cos 2\theta,$$

$$R=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)\cos(2\theta+2\beta)$$
$$=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)(\cos 2\theta\cos 2\beta-\sin 2\theta\sin 2\beta),$$

$$S=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)\cos(2\theta+2\gamma)$$
$$=\frac{1}{2}(A+B)+\frac{1}{2}(A-B)(\cos 2\theta\cos 2\gamma-\sin 2\theta\sin 2\gamma).$$

These are three simple equations for finding $\frac{1}{2}(A+B)$,

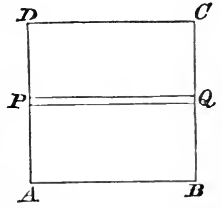
$\frac{1}{2}(A-B)\cos 2\theta$, and $\frac{1}{2}(A-B)\sin 2\theta$; thus we can deduce the value of $\tan 2\theta$, and thence the value of $\theta$; and then we shall know $A-B$ and $A+B$, and finally $A$ and $B$.

When $Q$, $R$, $S$ are not all equal we shall obtain one system of principal axes; when $Q$, $R$, $S$ are all equal, the values of $\tan 2\theta$ will be indeterminate, and any pair of lines at right angles to each other will be principal axes.

245. Hence if two different plane figures have the same moment of inertia about three axes through a common point, they will have the same principal axes at the point, and the same moment of inertia for every axis through the point. And if the plane figures have also the same mass and the same centre of gravity they will have the same moment of inertia about any straight line in the plane or at right angles to it: see Arts. 235 and 233. The term *plane figure* here includes any collection of particles which are all in one plane.

We shall now determine the value of the moment of inertia for various special cases.

246. *To determine the moment of inertia of a rectangle about an edge.*




Let $ABCD$ be a rectangle; it is required to determine the moment of inertia of the rectangle round the edge $AB$. Let $BC=b$; divide $BC$ into $n$ equal parts, and suppose straight lines drawn through the points of division parallel to $AB$. Thus $ABCD$ will be divided into $n$ equal strips; let $PQ$ represent the $r^{\text{th}}$ strip counting from $AB$.

If $M$ denote the mass of the rectangle $\frac{M}{n}$ will denote the

mass of $PQ$; and if $n$ is very great the distance of every point in the strip from $AB$ may be taken to be $\frac{r}{n}b$; so that the moment of inertia of the strip about $AB$ will be $\frac{M}{n} \times \frac{r^2b^2}{n^2}$. Hence the moment of inertia of the whole rectangle will be equal to the value when $n$ is made indefinitely great of the expression

$$\frac{Mb^2}{n^3}\{1^2+2^2+3^2+\ldots+n^2\}.$$

But, by Algebra, $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$;

thus the above expression is equal to

$$\frac{Mb^2}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right),$$

and when $n$ is made indefinitely great this becomes $\frac{Mb^2}{3}$.

247. The method of the preceding Article is substantially the same as that which is used in the higher parts of the subject for finding moments of inertia, but by the aid of the notation and the principles of the Integral Calculus the process is rendered shorter. It would be inconsistent with the plan of the present work to give any great attention to such investigations; we will however notice an indirect method of obtaining the result of the preceding Article which may admit of application in other cases.

248. Suppose two rectangles having the common edge $AB$; let $M$ denote the mass of one, and $b$ the length at right angles to $AB$; let $M'$ and $b'$ denote the corresponding quantities for the other. Then we shall shew that about the common axis $AB$

$$\frac{\text{Moment of inertia of first rectangle}}{\text{Moment of inertia of second rectangle}}=\frac{Mb^2}{M'b'^2}.$$

For, divide the second rectangle into the same number of very slender strips as the first, in the manner of Art. 246. Let $\mu$ denote the mass of a strip of the first

rectangle, and $\mu'$ the mass of the corresponding strip of the second; let $x$ denote the distance of the strip of the first rectangle from $AB$, and $x'$ the distance of the corresponding strip of the second: then it is obvious that

$$\frac{\mu}{\mu'}=\frac{M}{M'}, \text{ and } \frac{x^2}{x'^2}=\frac{b^2}{b'^2}.$$

Therefore
$$\frac{\mu x^2}{\mu' x'^2}=\frac{Mb^2}{M'b'^2}.$$

Since this relation holds for every pair of corresponding strips we obtain the result which had to be established. From the nature of the result it is easily understood and remembered; and the same remark will apply to various similar cases which will occur as we proceed. We shall now apply this to the problem of Art. 246.

249. *To find the moment of inertia of a rectangle about an edge.*

It follows from Art. 248 that the moment of inertia varies as the product of the mass into the square of the length of the rectangle. Let $M$ denote the mass, and $b$ the length; then the moment of inertia will be $\lambda Mb^2$, where $\lambda$ denotes some quantity which does not change when $b$ changes or when $M$ changes. Thus if there be another rectangle of mass $M'$ and length $b'$, having the same edge $AB$ as the former, then the moment of inertia of this rectangle about $AB$ will be $\lambda M'b'^2$. Hence the moment of inertia of the difference of the two rectangles about $AB$ will be $\lambda M'b'^2-\lambda Mb^2$: we will denote this by $Q$. Put $a$ for $AB$, and let $\tau$ denote the infinitesimal thickness of the rectangles. We suppose both of the rectangles to be of the same substance, so that the masses may be represented by the volumes; thus we may put

$$M=ab\tau, \qquad M'=ab'\tau;$$

therefore

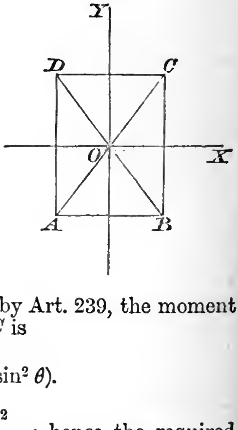
$$Q=\lambda\tau a\,(b'^3-b^3)=\lambda\tau a\,(b'-b)\,(b'^2+b'b+b^2)$$
$$=\lambda\mu\,(b'^2+b'b+b^2),$$

where $\mu$ denotes the difference of the masses of the two

rectangles, that is the mass of the body which has $Q$ for its moment of inertia round $AB$. This result is true whatever may be the values of $b'$ and $b$. But when the difference between $b'$ and $b$ is infinitesimal the body which has $Q$ for its moment of inertia becomes a strip, every particle of which may be considered to be at the distance $b$ from $AB$, so that the moment of inertia is $\mu b^2$. Hence when $b'=b$ we must have $\lambda(b'^2+b'b+b^2)=b^2$; therefore $\lambda=\frac{1}{3}$.

250. *To find the moment of inertia of a rectangle about a diagonal.*

Let $ABCD$ be the rectangle; let $AB=a$, and $BC=b$; and let $M$ denote the mass of the rectangle. Let the diagonals $AC$ and $BD$ intersect at $O$. Draw $OX$ parallel to $AB$, and $OY$ parallel to $BC$: let $\theta$ denote the angle $COX$. The moment of inertia of the rectangle about $OX$ by Arts. 233 and 246 is $M\frac{b^2}{3}-M\left(\frac{b}{2}\right)^2$, that is $M\frac{b^2}{12}$. Similarly the moment of inertia of the rectangle about $OY$ is $M\frac{a^2}{12}$. Now $OX$ and $OY$ are *principal axes* of the rectangle at $O$ by Art. 241; therefore, by Art. 239, the moment of inertia of the rectangle about $OC$ is




$$\frac{M}{12}(b^2\cos^2\theta+a^2\sin^2\theta).$$

But $\cos^2\theta=\frac{a^2}{a^2+b^2}$ and $\sin^2\theta=\frac{b^2}{a^2+b^2}$; hence the required moment of inertia is $\frac{M}{6}\cdot\frac{a^2b^2}{a^2+b^2}$.

251. Let $h$ denote the length of the perpendicular from $D$ on $AC$; then $h \times AC = AD \times DC$, for each expresses twice the area of the triangle $ADC$: thus $h = \frac{ab}{\sqrt{(a^2+b^2)}}$. Therefore the moment of inertia of a rectangle about a diagonal is $M\frac{h^2}{6}$, where $M$ denotes the mass, and $h$ the perpendicular on the diagonal from an opposite angle.

252. The moments of inertia of $ABC$ and $ADC$ about $AC$ are equal; for these two triangles are equal and are symmetrically situated with respect to $AC$. Thus the moment of inertia of each of them about $AC$ is half the moment of inertia of the rectangle $ABCD$; and therefore if $M$ now denote the mass of $ACD$ the moment of inertia of $ACD$ about $AC$ is $M\frac{h^2}{6}$.

253. If $M$ denote the mass of any triangle and $h$ the perpendicular from the vertex on the base the moment of inertia of the triangle about the base is $M\frac{h^2}{6}$. This is shewn in the preceding Article for the case in which the angle opposite to the base is a *right* angle, and it may be readily extended to the case in which this condition does not hold. For suppose $S$ to denote any point in $AC$ or $AC$ produced, and join $DS$, thus forming a triangle $ADS$. Then by drawing straight lines parallel to $AC$ we can divide the triangle $ADC$ and also the triangle $ADS$ into the same number of strips parallel to $AC$; the particles in each strip may be considered to be at the same distance from $AC$; and thus the moments of inertia of the strips will be in the same proportion as the masses of the strips, that is as the lengths of the strips, that is as $AC$ is to $AS$. Since this is true for each pair of strips it will be true for the whole triangles; and thus the required result is obtained.

254. The moment of inertia of any indefinitely thin wire bent into the form of a circle, about a straight line through the centre of the circle at right angles to its

plane is $Mr^2$; where $M$ denotes the mass and $r$ the radius of the circle. For every particle of the ring thus formed is at the same distance $r$ from the axis about which the moment of inertia is required. Hence the moment of inertia of such a ring about a diameter is $M\frac{r^2}{2}$; for the moment of inertia must be the same about any diameter, and the sum of the moments of inertia about two diameters at right angles to each other is equal to $Mr^2$ by Art. 235.

255. *To find the moment of inertia of a circle about an axis through its centre at right angles to its plane.*

It may be shewn as in Art. 248 that the moment of inertia must vary as the product of the mass into the square of the radius. Let $M$ denote the mass, and $r$ the radius; then the moment of inertia will be $\lambda Mr^2$ where $\lambda$ is some quantity which does not change when $r$ changes or when $M$ changes. Thus if there be a concentric circle of mass $M'$ and radius $r'$, its moment of inertia about the axis will be $\lambda M'r'^2$. Hence the moment of inertia of the difference of the two circles about the axis will be $\lambda M'r'^2 - \lambda Mr^2$: we will denote this by $Q$. Let $\tau$ denote the infinitesimal thickness of the circles; then representing the masses by the volumes, as in Art. 249, we may put

$$M = \pi r^2\tau, \qquad M' = \pi r'^2\tau;$$

therefore $Q = \lambda\pi\tau(r'^4 - r^4) = \lambda\pi\tau(r'^2 - r^2)(r'^2 + r^2) = \lambda\mu(r'^2 + r^2)$, where $\mu$ denotes the difference of the masses of the two circles, that is the mass of the body which has $Q$ for its moment of inertia round the axis considered. This result is true whatever may be the values of $r'$ and $r$. But when the difference between $r'$ and $r$ is infinitesimal the body becomes a ring every particle of which may be considered to be at the distance $r$ from the axis, so that the moment of inertia is $\mu r^2$. Hence when $r' = r$ we must have $\lambda(r'^2 + r^2) = r^2$; therefore $\lambda = \frac{1}{2}$: thus the required moment of inertia is $M\frac{r^2}{2}$.

256. The moment of inertia of a circle about a diameter is $M\frac{r^2}{4}$. For the moment of inertia must be the same about any diameter; and the sum of the moments of inertia about two diameters at right angles to each other is equal to $M\frac{r^2}{2}$ by Art. 235. The moment of inertia of a circle about any chord at the distance of $h$ from the centre will be $M\left(\frac{r^2}{4}+h^2\right)$ by Art. 233.

257. If $a$ and $b$ denote the lengths of the edges of a rectangle, and $M$ the mass, the moment of inertia about the former edge is $M\frac{b^2}{3}$, and about the latter $M\frac{a^2}{3}$. Hence, by Art. 235 the moment of inertia about an axis through a corner of the rectangle at right angles to its plane is $\frac{M}{3}(a^2+b^2)$.

258. If the moment of inertia of a body of mass $M$ about any axis is equal to $Mk^2$, then $k$ is called the radius of gyration of the body about the assigned axis. Thus, for example, the moment of inertia of a circle of radius $r$ about an axis through its centre at right angles to its plane is $M\frac{r^2}{2}$; so that $k^2=\frac{r^2}{2}$, and therefore the radius of gyration about this axis is $\frac{r}{\sqrt{2}}$.

259. Hitherto we have restricted ourselves as we stated in Art. 234, to the case of *plane figures;* we shall now shew by two examples that the results which we have thus obtained may be used to determine the moments of inertia of bodies which are not indefinitely thin. Let there be a rectangular parallelepiped of mass $M$; and let $a$, $b$, $c$ be the lengths of three edges which meet at a point: then the moment of inertia about the edge of length $c$ is $\frac{M}{3}(a^2+b^2)$. For suppose the body cut up into an indefinitely large number of indefinitely thin slices by planes at right angles to the axis; then, by Art. 257, the moment of inertia of

each slice is the product of its mass into $\frac{1}{3}(a^2+b^2)$: hence the moment of inertia of the whole mass is the product of that mass into $\frac{1}{3}(a^2+b^2)$. Next let there be a right circular cylinder of mass $M$, and let $r$ be the radius of the cylinder: then the moment of inertia of the cylinder about its axis of figure is $M\frac{r^2}{2}$. For we may suppose the cylinder cut up into slices as in the preceding Example; and then the result follows by the aid of Art. 255.

## Examples. XIX.

1. Shew that the moment of inertia of a triangle about the straight line which is drawn from an angle to the middle point of the opposite side is $M\frac{p^2}{6}$; where $M$ is the mass of the triangle, and $p$ the perpendicular drawn to this straight line from either end of the side.

2. Let $ABC$ be a triangle; let $D$, $E$, $F$ be the middle points of the sides opposite to $A$, $B$, $C$ respectively; and denote by $M$ the mass of the triangle: shew that the moment of inertia of the triangle about $AD$ is the same as that of two particles each of the mass $\frac{M}{12}$ placed at $B$ and $C$ respectively: or of two particles each of the mass $\frac{M}{3}$ placed at $E$ and $F$ respectively.

3. Shew that the moment of inertia of a triangle of mass $M$ about any axis in its plane is the same as that of three particles each of mass $\frac{M}{3}$ at the middle points of the sides.

4. Shew that the moment of inertia of a triangle of mass $M$ about any axis in its plane is the same as that of three particles each of mass $\frac{M}{12}$ at the angular points, and a particle of mass $\frac{3M}{4}$ at the centre of gravity.

5. Shew that the moment of inertia of a triangle of mass $M$ about any axis at right angles to its plane is the same as that of three particles each of mass $\frac{M}{3}$ at the middle points of the sides.

6. Shew that the moment of inertia of a triangle of mass $M$ about any axis at right angles to its plane is the same as that of three particles each of mass $\frac{M}{12}$ at the angular points, and a particle of mass $\frac{3M}{4}$ at the centre of gravity.

7. Find the moment of inertia of a triangle about an axis through its centre of gravity at right angles to its plane.

8. Find the moment of inertia of a rectangular parallelepiped about an axis through the centre of gravity parallel to an edge.

9. A given mass is to be formed into a plane figure so as to have the least moment of inertia about an axis passing through a given point, at right angles to the plane of the figure: shew that the figure must be a circle having its centre at the given point.

10. If at a given point of a plane figure the moments of inertia are equal about two straight lines unequally inclined to the principal axes, the moments of inertia about the two principal axes through that point must be equal.

## XX. *Motion round a fixed axis.*

260. We have already introduced the term *angular velocity* in the particular case in which this velocity is constant; see Art. 171: we shall now have to consider it more generally. When a particle describes a circle the straight line drawn from the particle to the centre describes an angle. The rate at which the angle is described is called *angular velocity*, and this may be uniform or variable. When the angular velocity is variable it may be considered as uniform for an indefinitely short time; and thus, as in Art. 171, we have a relation between the *angular velocity* of the describing straight line and the *linear velocity* of its extremity at any instant, namely this: the *linear velocity is the product of the angular velocity into the radius.*

261. Corresponding to the term *velocity* we have the term *acceleration;* this denotes the amount of velocity which is added in a unit of time if the velocity is increased uniformly, or the velocity which would be added in a unit of time if the rate of increase were to continue for a unit of time what it is at the instant: see Arts. 18 and 19. In like manner corresponding to the term *angular velocity* we have the term *angular acceleration;* this denotes the amount of angular velocity which is added in a unit of time if the angular velocity is increased uniformly, or the angular velocity which would be added in a unit of time if the rate of increase were to continue for a unit of time what it is at the instant. And from the conclusion of the last Article it follows that the *linear acceleration in the direction of the tangent is the product of the angular acceleration into the radius.*

262. When a body turns round a fixed axis every point of it describes a circle or an arc of a circle in a plane at right angles to the axis. The angle which the radius belonging to one point describes is equal to the angle which the radius belonging to any other point describes in the same time. Thus the *angular velocity* is the same for every point of the body, and the *angular acceleration* is

the same for every point of the body. It is the object of the present Chapter to shew how the *angular acceleration* is to be determined; and when this is known the *angular velocity* may be sought, and finally the angle through which the body has turned from any assigned position.

263. *To investigate the angular acceleration of a body which can turn round a fixed axis and is acted on by given forces.*

Let $m_1, m_2, m_3, \ldots$ denote the masses of the particles of the body; and let $r_1, r_2, r_3, \ldots$ denote the radii of the corresponding circles which they can describe. Let $\omega$ denote the angular velocity at any instant, and $\chi$ the angular acceleration. Then the *effective* forces corresponding to $m_1$ are $m_1 r_1 \omega^2$ along the radius towards the centre of the circle which this particle can describe, and $m_1 r_1 \chi$ along the tangent: see Art. 227. Suppose the *impressed* forces to consist of $P_1$ acting at an arm $p_1$, $P_2$ acting at an arm $p_2$, $P_3$ acting at an arm $p_3$, and so on; each force being supposed to be in some plane at right angles to the axis. By D'Alembert's Principle the impressed forces and the effective forces reversed must satisfy the condition of statical equilibrium; hence by Arts. 100 and 104 of the *Statics* the sum of the moments round the axis must vanish. Now the moment of such a force as $m_1 r_1 \omega^2$ is zero, because the direction of the force passes through the axis; and the moment of such a force as $m_1 r_1 \chi$ is $m_1 r_1 \chi \times r_1$, because the direction of the force is at right angles to the radius $r_1$. Therefore

$$P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots - m_1 r_1^2 \chi - m_2 r_2^2 \chi - m_3 r_3^2 \chi \ldots = 0\,;$$

This equation may be written

$$\chi = \frac{P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots}{m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots},$$

that is, according to a mode of abbreviation already used,

$$\chi = \frac{\Sigma Pp}{\Sigma m r^2}.$$

264. In Art. 87 we obtained a result which may be expressed verbally thus: when force acts on a free body

the acceleration is equal to the quotient of the force by the mass. If we use the word *inertia* as synonymous with *mass*, which some of the old writers did, we may say that the acceleration is equal to the quotient of the force by the inertia. In the formula just obtained for $\chi$ we have in the numerator the *moment of the forces*, and hence the name *moment of inertia* might be given by analogy to the denominator, so that we may have this result: when a body can turn round a fixed axis the angular acceleration is the quotient of the moment of the forces by the moment of inertia about the axis. It is obvious that the term *moment of inertia* now has precisely the same sense as it had in Art. 232.

265. In the Chapters on Dynamics which we have given we have mainly confined ourselves to cases in which the acceleration in the direction of the motion is *constant;* the reason for this is that the mathematical difficulties in other cases are greater than the student at this stage can be assumed able to overcome. The example of Art. 151 shews how complex the investigation may be of even simple problems which do not fall within the restriction we have named.

266. As an illustration of Art. 263 we will now discuss the motion of a system of the nature of the Wheel and Axle: see the figure in Art. 180 of the *Statics*. Suppose there to be a body of mass $\mu_1$ and therefore of weight $\mu_1 g$, which acts at an arm $a_1$; let this be rising. Suppose there to be a body of mass $\mu_2$ and therefore of weight $\mu_2 g$, which acts at an arm $a_2$; let this be descending. Let the mass of the body which forms the Wheel and Axle be $M$, and let its moment of inertia about the axis be $Mk^2$. Let $T_1$ be the tension of the string attached to the mass $\mu_1$, and $T_2$ the tension of the string attached to the mass $\mu_2$; then if $\chi$ denote the angular acceleration of the body which forms the Wheel and Axle we have, by Art. 263,

$$Mk^2\chi = T_2 a_2 - T_1 a_1 \dots\dots\dots\dots\dots\dots(1).$$

The axis is supposed to pass through the centre of gravity of the body which forms the Wheel and Axle, so that the moment of the weight of the body round the

axis vanishes. If the system were in equilibrium the values of $T_1$ and $T_2$ would be known; namely the former would be equal to $\mu_1 g$, and the latter to $\mu_2 g$: but these will not be the values when there is motion, and we shall proceed to find two other equations by the aid of which we can eliminate $T_1$ and $T_2$. The velocity of the descending weight is at any instant the same as that of the circumference of the circle to which its string is attached, so that the acceleration is $a_1\chi$: therefore, by Art. 87,

$$\mu_1 a_1 \chi = T_1 - \mu_1 g \qquad (2).$$

Similarly from considering the descending weight we have

$$\mu_2 a_2 \chi = \mu_2 g - T_2 \qquad (3).$$

Substitute in (1) the values of $T_1$ and $T_2$ from (2) and (3);

thus $\quad Mk^2\chi = a_2(\mu_2 g - \mu_2 a_2 \chi) - a_1(\mu_1 a_1 \chi + \mu_1 g)$,

therefore $\quad (Mk^2 + \mu_1 a_1^2 + \mu_2 a_2^2)\chi = (\mu_2 a_2 - \mu_1 a_1) g$.

Thus the angular acceleration is *constant*, namely equal to

$$\frac{(\mu_2 a_2 - \mu_1 a_1) g}{Mk^2 + \mu_1 a_1^2 + \mu_2 a_2^2}.$$

Denote this by $\lambda$: then by the same processes as we have used in Arts. 36 and 37, we see that if the body which forms the Wheel and Axle starts with an angular velocity $\gamma$, the angular velocity at the end of the time $t$ will be $\gamma + \lambda t$; and the angle which the body has turned through in the time $t$ will be $\gamma t + \frac{1}{2}\lambda t^2$.

267. *To investigate the motion of a heavy body which can turn round a fixed horizontal axis.*

Let $M$ denote the mass of the body, $h$ the length of the perpendicular from the centre of gravity of the body on the axis, $\theta$ the angle which this straight line makes at any instant with the horizontal plane through the axis. Then the weight of the body is $Mg$, and the moment of this round the axis at the instant considered is $Mgh\cos\theta$. Therefore by Art. 263 the angular acceleration at the instant

$$= \frac{Mgh\cos\theta}{\Sigma mr^2}.$$

Now let $Mk^2$ denote the moment of inertia of the body about a straight line through the centre of gravity parallel to the axis; then, by Art. 233, $\Sigma mr^2 = M(h^2 + k^2)$; therefore the angular acceleration $= \dfrac{gh \cos \theta}{h^2 + k^2}$.

Now suppose that instead of the heavy *body* we had a heavy *particle* attached to the axis by a rigid rod without weight of the length $l$, thus forming what is called a *simple pendulum;* we shall find, on investigating the motion by the principles applied to the case of the heavy body, that the angular acceleration when the rod is inclined at an angle $\theta$ to the horizon is $\dfrac{lg \cos \theta}{l^2}$, that is $\dfrac{g \cos \theta}{l}$. Hence by comparing the two cases we arrive at the following important result: the motion of the heavy body is the same as that of a simple pendulum of length $l$ where $l = \dfrac{h^2 + k^2}{h}$. Therefore the results already obtained with respect to the simple pendulum in Art. 151 will hold for the heavy body. Thus if $\omega$ be the angular velocity acquired from rest while $\theta$ changes from $90^0 - \alpha$ to $90^0 - \beta$ we shall have

$$(l\omega)^2 = 2gl (\cos \beta - \cos \alpha);$$

and the time of a small oscillation will be $\pi \sqrt{\dfrac{l}{g}}$.

268. Thus we see that the motion of a heavy body round a fixed horizontal axis is exactly the same as that of a certain simple pendulum; this pendulum is called the *equivalent simple pendulum.* Let a straight line be drawn at right angles to the axis so as to pass through the centre of gravity of the heavy body, and let it be produced till its length measured from the axis is equal to the length of the equivalent simple pendulum; then the extremity of this straight line is called the *centre of oscillation* of the body.

269. The most important property connected with the centre of oscillation is that which is briefly stated by saying that the *centres of oscillation and suspension are convertible:*

the meaning of this statement will appear from the investigation to which we now proceed. Let the distance of the centre of gravity from the centre of oscillation be denoted by $h'$. Suppose the body to be put in motion round an axis through the centre of oscillation parallel to the original axis, instead of round the original axis; then the length of the equivalent simple pendulum in this case will be $\frac{h'^2+k^2}{h'}$.

But $h'=\frac{k^2+h^2}{h}-h=\frac{k^2}{h}$; and $\frac{k^2+h'^2}{h'}=\frac{k^2}{h'}+h'=h+\frac{k^2}{h}=\frac{h^2+k^2}{h}$; and this is the length of the original equivalent simple pendulum. Thus the motion round the new axis will be precisely the same as the motion round the original axis: for instance, the time of a small oscillation in either case is $\pi\sqrt{\frac{l}{g}}$, where $l$ stands for $\frac{h^2+k^2}{h}$.

270. We have $\frac{h^2+k^2}{h}=\left(\frac{k}{\sqrt{h}}-\sqrt{h}\right)^2+2k$. Now suppose the body to be put in motion in succession round various axes which are all *parallel;* then $k$ remains the same in all these cases while $h$ may vary: and we see that the *least* value of the length of the equivalent simple pendulum is when $\frac{k}{\sqrt{h}}-\sqrt{h}$ vanishes, that is when $h=k$.

## EXAMPLES. XX.

1. Find the length of the equivalent simple pendulum when a rectangle turns round an edge which is horizontal.

2. Also when a cube turns round an edge which is horizontal.

3. Also when an equilateral triangle turns round a horizontal axis through an angular point at right angles to its plane.

4. A circular arc turns round a horizontal axis through its middle point at right angles to the plane of the arc: shew that the length of the equivalent simple pendulum is equal to the diameter of the circle.

## XXI. *Miscellaneous Theorems.*

271. In Chapter XIX. we confined ourselves to the case of *plane figures*, and easy deductions from this case; we now proceed to the more general problem, namely that in which the moment of inertia of any body is considered. Through any point $O$ draw three straight lines $OX$, $OY$, $OZ$ mutually at right angles, like the edges of a cube which meet at a common point. Let $A$, $B$, $C$ denote the moments of inertia of an assigned body about $OX$, $OY$, $OZ$ respectively. Through $O$ draw any straight line $OL$; let $Q$ denote the moment of inertia of the body about $OL$: we shall now connect $Q$ with $A$, $B$, $C$. Let $\alpha$, $\beta$, $\gamma$ denote the angles which $OL$ makes with $OX$, $OY$, $OZ$ respectively. Suppose one particle of the body of mass $m$ to be at a point $T$; let $x$, $y$, $z$ denote the perpendiculars from $T$ on the planes $YOZ$, $ZOX$, $XOY$ respectively; and let $r$ denote $OT$. The perpendicular from $T$ on $OL = r \sin TOL$; the square of this $= r^2 \sin^2 TOL = r^2 - r^2 \cos^2 TOL$: and, by Art. 269 of the *Statics*, this

$$= r^2 - r^2 (\cos TOX \cos LOX + \cos TOY \cos LOY + \cos TOZ \cos LOZ)^2$$

$$= r^2 - (x \cos \alpha + y \cos \beta + z \cos \gamma)^2$$

$$= (x^2 + y^2 + z^2)(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - (x \cos \alpha + y \cos \beta + z \cos \gamma)^2$$

$$= (y^2 + z^2) \cos^2 \alpha + (z^2 + x^2) \cos^2 \beta + (x^2 + y^2) \cos^2 \gamma - 2yz \cos \beta \cos \gamma - 2zx \cos \gamma \cos \alpha - 2xy \cos \alpha \cos \beta.$$

Then $Q = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma$

$$- 2 \cos \beta \cos \gamma \, \Sigma myz - 2 \cos \gamma \cos \alpha \, \Sigma mzx - 2 \cos \alpha \cos \beta \, \Sigma mxy.$$

272. The quantity $\Sigma myz$ is called the *product of inertia* of the body for the axes $OY$ and $OZ$; and similar names are given to $\Sigma mzx$ and $\Sigma mxy$. If at a point in a body three straight lines can be drawn at right angles to each other,

such that the three products of inertia of the body all vanish, the three straight lines are called *principal axes* of the body at the point.

273. We shall now shew that at every point of a body principal axes exist. The moment of inertia of an assigned body about an assigned axis is by definition a finite positive quantity. Thus among all the axes which pass through a given point there must be one for which the moment of inertia is greater than it is for all other axes, or at least for which it is as great as it is for any other axis. Let $O$ denote the point considered, and suppose that $OX$ is an axis for which the moment of inertia is as great as it is for any axis through $O$. Then, with the notation of Art. 271, it follows that $A - Q$ cannot be negative. Now

$$A - Q = A(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - Q$$
$$= (A - B)\cos^2\beta + (A - C)\cos^2\gamma$$
$$+ 2\cos\beta\cos\gamma\,\Sigma myz + 2\cos\gamma\cos\alpha\,\Sigma mzx + 2\cos\alpha\cos\beta\,\Sigma mxy.$$

Suppose now that $OL$ is in the plane $XOY$, so that $\gamma = 90^0$, and therefore $\cos\gamma = 0$; then

$$A - Q = (A - B)\cos^2\beta + 2\cos\alpha\cos\beta\,\Sigma mxy;$$

and since $OL$ is in the plane $XOY$ we have $\alpha + \beta = 90^0$; thus

$$A - Q = \sin^2\alpha\,(A - B + 2\cot\alpha\,\Sigma mxy).$$

Then proceeding as in Art. 239 we see that $\Sigma mxy$ must be zero. In precisely the same way, by supposing $OL$ to be in the plane $XOZ$ we see that $\Sigma mxz$ must be zero. Hence by taking for $OX$ what we may call the axis of *greatest* moment of inertia it follows that the two products of inertia $\Sigma mxy$ and $\Sigma mxz$ vanish.

We have not yet fixed the position of the axes $OY$ and $OZ$ in the plane in which they must lie. Let then $OZ$ be such that $C$ is the least moment of inertia for all axes in this plane, or at least as small as any. Suppose $OL$ to be in the plane $YOZ$, so that $\cos\alpha = 0$, and $\beta + \gamma = 90^0$.

Then $\qquad Q = B\cos^2\beta + C\cos^2\gamma - 2\cos\beta\cos\gamma\,\Sigma myz,$

and $\qquad Q - C = (B - C)\cos^2\beta - 2\cos\beta\cos\gamma\,\Sigma myz$

$$= \sin^2\gamma\,(B - C - 2\cot\gamma\,\Sigma myz).$$

This expression cannot be negative, and hence in the manner of Art. 239 it follows that $\Sigma myz$ must be zero.

Thus the existence of principal axes at any point $O$ is established. And if the moments of inertia about these are known the moment of inertia about any other axis is known from the formula

$$Q = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma.$$

We may observe that the least of the three $A$, $B$, $C$ will also be the least moment of inertia about all axes through $O$. For

$$Q - C = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - C(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$
$$= (A - C)\cos^2\alpha + (B - C)\cos^2\beta\,;$$

and this cannot be negative if $C$ be not greater than either $A$ or $B$.

274. The position of axes for which the three products of inertia vanish, that is the position of principal axes, is often sufficiently obvious. Suppose, for example, that we require the position of principal axes for a right circular cone at the vertex: take the axis of figure and any two straight lines at right angles to each other and to this, and they will constitute principal axes. For suppose $OX$ to coincide with the axis of figure; then $\Sigma mxy$ vanishes. For consider a particle of mass $m$ at a point of which the co-ordinates are $x$ and $y$; it is plain that there is also a corresponding particle of mass $m$ at the point of which the co-ordinates are $x$ and $-y$. Hence we see that $\Sigma mxy$ consists of terms which may be arranged in pairs, so that the two terms in a pair are numerically equal but of opposite signs; and therefore $\Sigma mxy$ vanishes. Similarly $\Sigma mxz$ and $\Sigma myz$ vanish.

275. The values of the products of inertia at any point may be made to depend on the values of the products of inertia for parallel axes through the centre of gravity. For as in Art. 242 we can shew that

$$\Sigma mxy = \Sigma mx'y' + hk\,\Sigma m\,;$$

and that similar equations hold for the other two products of inertia.

276. *The sum of the moments of inertia of a given body about any three axes at right angles to each other passing through a given point is constant.*

Let $O$ denote the given point; and $OX$, $OY$, $OZ$ three straight lines at right angles to each other. Let $m$, $x$, $y$, $z$ have the same meaning as in Art. 271. Then the moment of inertia about $OX$ is $\Sigma m(y^2+z^2)$, that about $OY$ is $\Sigma m(z^2+x^2)$, and that about $OZ$ is $\Sigma m(x^2+y^2)$. The sum of these three expressions is $2\Sigma m(x^2+y^2+z^2)$, that is $2\Sigma mr^2$, where $r$ is the distance of the particle of mass $m$ from $O$. And $\Sigma mr^2$ has obviously the same value whatever may be the position of $OX$, $OY$, $OZ$.

277. *To find the moment of inertia of an indefinitely thin spherical shell about a diameter of the sphere.*

Let $M$ denote the mass of the shell, $a$ the radius of the sphere, $Q$ the required moment of inertia. It is plain that the value of $Q$ is the same for *any* diameter; hence, by Art. 276, we have $3Q=2\Sigma mr^2=2Mr^2$: therefore $Q=M\dfrac{2r^2}{3}$.

278. *To find the moment of inertia of a sphere about a diameter.*

It may be shewn as in Art. 248 that the moment of inertia must vary as the product of the mass into the square of the radius. Let $M$ denote the mass, and $r$ the radius, then the moment of inertia will be $\lambda Mr^2$, where $\lambda$ is some quantity which does not change when $r$ changes or when $M$ changes. Thus if there be a concentric sphere of mass $M'$ and radius $r'$, its moment of inertia about the diameter will be $\lambda M'r'^2$. Hence the moment of inertia of the difference of the two spheres about the diameter will be $\lambda(M'r'^2-Mr^2)$. Let $\mu$ denote the mass of the difference of the spheres, and $\mu k^2$ its moment of inertia about the diameter; so that

$$\mu k^2=\lambda(M'r'^2-Mr^2).$$

Now it is known that the volume of a sphere is the

product of $\frac{4\pi}{3}$ into the cube of the radius; therefore representing the masses by the volumes, as in Art. 249, we have

$$M=\frac{4\pi}{3}r^3, \quad M'=\frac{4\pi}{3}r'^3, \quad \mu=\frac{4\pi}{3}(r'^3-r^3).$$

Hence $$\frac{4\pi}{3}(r'^3-r^3)k^2=\frac{4\pi}{3}\lambda(r'^5-r^5),$$

so that $$k^2=\frac{\lambda(r'^4+r'^3r+r'^2r^2+r'r^3+r^4)}{r'^2+r'r+r^2}.$$

Now, this, being always true, holds when the difference between $r'$ and $r$ is infinitesimal; and then by Art. 277 we have $k^2=\frac{2r^2}{3}$: thus $\frac{2r^2}{3}=\lambda\frac{5r^2}{3}$, so that $\lambda=\frac{2}{5}$.

279. We may propose questions relative to the effect of *impulsive* forces on a body which can turn round a fixed axis; the discussion of such questions will resemble that which we have given in Chapters IX. and X. with respect to the subject of the Collision of Bodies.

280. *To determine the change of motion produced by impulsive forces acting on a body which can turn round a fixed axis.*

Let $m_1, m_2, m_3,\ldots$ denote the masses of the particles of the body; let $r_1, r_2, r_3,\ldots$ denote the respective distances of the particles from the fixed axis; and suppose that $\omega$ is the angular velocity with which the body is turning round the axis just before the impulses. Let $Q_1, Q_2, Q_3,\ldots$ denote impulsive forces which act simultaneously on the body, at distances $q_1, q_2, q_3,\ldots$ respectively from the axis, each force being supposed to be in some plane at right angles to the axis. Suppose that the angular velocity is thus suddenly changed to $\omega'$. Then the *effective impulsive force* on $m_1$ is $m_1r_1(\omega'-\omega)$ in the direction at right angles to that of $r_1$; and similar expressions hold for the other particles. By D'Alembert's Principle and Arts. 100 and 104 of the *Statics* we have

$$Q_1q_1 + Q_2q_2 + Q_3q_3 + \dots - (m_1r_1{}^2 + m_2r_2{}^2 + m_3r_3{}^2 + \dots)(\omega' - \omega) = 0.$$

We may write this result thus,

$$\omega' - \omega = \frac{\Sigma Qq}{\Sigma mr^2};$$

that is the numerator of the fraction which gives $\omega' - \omega$ is the sum of the moments of the impulsive forces round the fixed axis, and the denominator is the moment of inertia of the body about that axis.

281. In such a case as the preceding there are forces which act on the body at the points where the axis is fixed; but these do not enter into the equation which gives $\omega' - \omega$, because they have no moment round the axis. If the direction of any one of the impulsive forces is not at right angles to the fixed axis, this force must be resolved into two, one parallel to the axis, and the other at right angles to it: the former component will have no effect on the angular velocity, the latter alone will occur in the equation.

282. Suppose a body at rest but capable of turning round a fixed axis: we may enquire whether it is possible to put the body in motion by a blow without producing any impulsive pressure on the axis. We shall confine ourselves to the case in which the blow is given in a plane which is at right angles to the axis and which divides the body *symmetrically*, so that we may assume that there can be no impulsive pressure on the axis out of this plane.

Let $Q$ denote the blow, $q$ the arm at which it acts, $\omega'$ the angular velocity given to the body. As there is to be no impulsive pressure on the axis the force $Q$ with the effective forces reversed must satisfy the conditions of equilibrium. Now, as in Art. 280, the effective force on the particle of mass $m_1$ is $m_1r_1\omega'$, the direction being at right angles to that of $r_1$; and similarly for the other particles. Let $G$ denote the centre of gravity of the body, and let the perpendicular from $G$ on the axis meet it at $O$.

As in Art. 97 of the Statics the system of effective forces reduces to a certain single force at $O$ and a couple. Then, as in Art. 278 of the Statics, the direction of the single force is at right angles to $OG$, and its magnitude is $\omega' OG\Sigma m$; and the moment of the couple is $\omega'\Sigma mr^2$. Hence this force and couple reversed must be in equilibrium with $Q$. Moreover $Q$ may be replaced by a force $Q$ at $O$ parallel to its original direction, and a couple of which the moment is $Qq$. Hence for the equilibrium required we must have

(1) The direction of $Q$ must be at right angles to $OG$.

(2) $Q$ must be equal to $\omega' OG\Sigma m$.

(3) $Qq$ must be equal to $\omega'\Sigma mr^2$.

From (2) and (3) by division we deduce

$$q = \frac{\Sigma mr^2}{OG\Sigma m}. \tag{4}$$

Hence finally $q$ must have the value determined by (4), and $Q$ must have the direction determined by (1): these are the necessary and sufficient conditions in order that the blow may produce no impulsive pressure on the axis.

283. It will be observed that the preceding investigation does not determine the *point* at which the blow must be given, but only the value of the perpendicular from the axis on the *direction* of the blow. This might have been anticipated from the fact that the effect of a force will be the same at whatever point of its line of action we suppose it applied. The point at which the direction of the blow meets $OG$ produced is called the *centre of percussion.* By comparing the value of $q$ with that given for the length of the *equivalent simple pendulum,* according to Art. 269, we see that the *centre of percussion* and the *centre of oscillation* relative to the same axis coincide.

## EXAMPLES. XXI.

1. Shew that the two systems mentioned in Example 3 of Chapter XIX. have the same moment of inertia for *any axis whatever.* Shew also that a similar statement may be made with respect to Examples 4, 5, and 6.

2. Shew that the moment of inertia of a rectangle of mass $M$ about any axis is the same as that of four particles each of mass $\frac{M}{12}$ at the angular points, and a particle of mass $\frac{2M}{3}$ at the centre of gravity.

3. Find the position of principal axes at the centre of gravity of a rectangular parallelepiped, and the moment of inertia about each of them.

4. Shew that the moment of inertia of a uniform straight rod of infinitesimal thickness about an axis through one end at right angles to the rod is the same as that of a particle at the other end equal in mass to one-third of the rod. Shew also that this is true for an axis through the end inclined at any angle to the rod.

5. Shew that the moment of inertia of a uniform straight rod of infinitesimal thickness of mass $M$ about any axis through one end is equal to that of a particle of mass $\frac{M}{6}$ at the other end, and a particle of mass $\frac{2M}{3}$ at the middle point.

6. Shew that the moment of inertia of a uniform straight rod of infinitesimal thickness of mass $M$ about any axis is equal to that of a particle of mass $\frac{M}{6}$ at each end, and a particle of mass $\frac{2M}{3}$ at the middle point.

## MISCELLANEOUS EXAMPLES.

1. A stone falls down 100 feet, determine the time of motion.

2. A stone falls down a well and is heard to strike the water after 3 seconds: find the depth of the well, supposing sound to be transmitted instantaneously.

3. If the space described in falling for 11 seconds from rest be 556·6 feet, find the acceleration.

4. A body, starting from a given point, moves vertically downwards at the rate of 32·2 feet per second. After four seconds a heavy body falls from the same point under the action of gravity. Shew that it will overtake the first body at a distance of 257·6 feet from the starting point.

5. Any number of smooth fixed straight rods, not in the same plane, pass through a given point; and a heavy particle slides down each rod, the particles starting simultaneously from the given point. If the rods be so situated that the particles are at one instant of their motion in the same plane, prove that they will be so throughout it, and that a circle can be described passing through them.

6. $AB$ is the vertical diameter of a sphere; a chord is drawn from $A$ meeting the surface at $P$, and the tangent plane at $B$ at $Q$: shew that the time down $PQ$ varies as $BQ$, and that the velocity acquired varies as $BP$.

7. Find a point at a given distance from the centre of a given vertical circle, such that the time of falling from it to the centre is less than the time of falling to any point on the circumference except one, and equal to the time of falling to this point.

8. Find the locus of points in a given vertical plane from which the times of descent down smooth Inclined Planes to a fixed point in the vertical plane vary as the length of the Planes.

9. A body is projected along a smooth horizontal table with a velocity $g$: find the length to which the table must be prolonged in the direction of the body's motion, so that the body after leaving the table may strike a point whose distances measured horizontally and vertically from the point of projection are $3g$ and $2g$ respectively.

10. A heavy particle is projected from a given point in a given direction so as to touch a given straight line: give

a geometrical construction for determining the point of contact and the elements of the path described. If the direction of projection be not fixed, find the path so that the velocity of projection may be the least possible.

11. A chord is drawn joining any point on the circumference of a vertical circle with the lowest point: shew that if a heavy body slide down this chord the parabola which it describes on leaving the chord has its directrix passing through the upper end of the chord.

12. Chords are drawn joining any point on the circumference of a vertical circle with the highest and lowest points; a heavy body slides down the lower chord: shew that the parabola which it will describe after leaving the chord is touched by the other chord, and that the locus of the points of contact is a circle.

13. A heavy body is projected from one fixed point so as to pass through another which is not in the same horizontal line with it: shew that the locus of the focus of its path is an hyperbola.

14. A force acting uniformly during one tenth of a second produces in a given body the velocity of one mile per minute: compare the force with the weight of the body.

15. One end of a string is fastened to a weight $P$; the string passes over a fixed Pully, and under a moveable Pully, and has its other end attached to a fixed point; a weight $Q$ is attached to the moveable Pully: determine the motion, supposing the three parts of the string all parallel.

16. In the formulæ of Art. 101 shew that if the velocities $u$ and $u'$ are each increased by the same quantity, so are the velocities $v$ and $v'$.

17. From the formulæ of Art. 101 determine the values of $v$ and $v'$ if $m=em'$; also if $m'=em$.

18. A body of given mass is moving in a given direction: determine the magnitude and the direction of a blow which will cause it to move with the same velocity in a direction at right angles to the former.

19. A projectile at the instant it is moving with the velocity $v$ at an inclination $\alpha$ to the horizon impinges on a vertical plane which makes an angle $\beta$ with the plane of motion of the projectile: find the velocity after impact.

20. Small equal spherical balls of perfect elasticity are

placed at the corners of a regular hexagon; one of them is projected with the velocity $u$, so as to strike all the others in succession and to pass through its original position: find the velocity with which it returns.

21. In the preceding Example shew that each of the five balls starts at right angles to an adjacent side of the hexagon; and find the velocity with which each starts.

22. Two perfectly elastic balls of equal mass impinge: shew that if the directions of motion after impact are parallel, the cosine of the angle between their original directions is equal to the ratio of the product of the velocities after impact to the product before impact.

23. Of two equal and perfectly elastic balls one is projected so as to describe a parabola, and the other is dropped from the directrix so as just to fall upon the first when at its highest point: determine the position of the vertex of the new parabola.

24. A mark in a vertical wall appears elevated at an angle $\beta$ at a certain point in a horizontal plane; from this point a ball is projected at the mark and after striking it returns to the point of projection: shew that if $\alpha$ be the angle of projection $\tan\alpha = (1+e)\tan\beta$.

25. A plane is inclined at an angle $\beta$ to the horizon; a particle is projected from a point in the plane at an inclination $\alpha$ to the horizon, with the velocity $u$, and the particle rebounds from the plane: find the time of describing $n$ parabolic arcs.

26. In the preceding Example find the condition which must hold in order that after describing $n$ parabolic arcs the particle should be again at the starting point.

27. A particle is projected with a given velocity at a given inclination to the horizon from a point in an inclined plane: find the whole time which elapses before the particle ceases to hop.

28. In the preceding Example find the condition which must hold in order that the particle may cease to hop just as it is again at the starting point.

29. In Example 25 find the cotangent of the inclination to the plane of the direction of motion of the particle at the beginning of the $(n+1)^{\text{th}}$ arc.

30. In Example 25 if the elasticity be perfect find the

condition which must hold in order that the particle may rise vertically at the $n^{\text{th}}$ rebound.

31. Shew that the time of descent to the lowest point of a very small circular arc is to the time of descent down its chord as the circumference of a circle is to four times its diameter.

32. If the resistance on similar steamers moving uniformly is proportional to the product of the transverse section and the square of the velocity, while their Horse-power is proportional to the tonnage, find how the velocity varies according to the tonnage.

33. A particle descends down a smooth circular tube of very small bore, and impinges on an equal particle at rest at the lowest point of the tube: if $h$ denote the vertical height through which the particle descends, determine the vertical height to which each particle will rise after impact.

34. If the weight attached to the free end of the string in a system of Pullies in which the same string passes round each of the Pullies be $m$ times that which is necessary to maintain equilibrium, shew that the acceleration of the ascending weight is $\frac{(m-1)g}{mn+1}$, where $n$ is the number of parts of the string at the lower block, and the grooves of the Pullies are supposed perfectly smooth. Compare the tension of the string with the ascending weight.

35. Two particles move with constant accelerations in given straight lines. If at any instant their relative velocities in any two directions are as their relative accelerations in the same directions, shew that the velocity of one particle always bears a constant ratio to the velocity of the other.

36. Two equal perfectly elastic balls are let fall at the same instant, one from the height $\frac{g}{2}$ and the other from the height $\frac{9g}{2}$ above a horizontal table; shew that at the end of $6n-1$ seconds the velocity of the centre of gravity changes suddenly from 0 to $g$, and at the end of $6n+1$ seconds the velocity of the centre of gravity changes suddenly from $g$ to 0.

# STATICS. ANSWERS.

I. 1. 8 lbs. 2. 32 inches. 3. $\frac{bP}{a}$ lbs. 4. $\frac{Q}{P}a$ inches. 5. 9 lbs. and 3 lbs.: 1 inch. 6. As 3 is to 4. 7. As 4 is to 3. 8. $\frac{7}{20}$ of a cubic foot. 9. As 16 is to 9. 10. As $\frac{m}{a}$ is to $\frac{n}{b}$.

II. 1. 64, 8. 2. 37. 3. 9, 12. 4. 3, 6. 6. $5\sqrt{2}$ lbs., at an angle of $45^0$ with the resultant. 8. As $\sqrt{3}$ is to 2. 9. A right angle. 10. 5, $5\sqrt{3}$. 11. In a straight line. 12. $120^0$. 13. The tension of the shorter string is 4 lbs., and of the longer string 3 lbs.

III. 2. By Art. 34, forces 1, 1, 1 are in equilibrium and may be omitted; thus the resultant is equivalent to that of forces 1 and 2 at an angle of $120^0$. 4. See Art. 34. 5. 15 lbs., 20 lbs. 6. 4 lbs. 7. Let $OA$ and $OB$ denote the equal forces, $OD$ their resultant; produce $AO$ to $C$ so that $OC=2OA$; and let $OE$ be the resultant of $OB$ and $OC$: then it is given that $OE=OD$. The resultant of $OE$ and $OD$ is equivalent to that of twice $OB$ and half $OC$, and is therefore equal to $OE$. 8. It follows from Example 7 that the angle $EOD=120^0$. 12. The resultant is $2\sqrt{2}$ lbs., and it is parallel to a side of the square. 13. The resultant coincides in direction with the straight line from the point to the intersection of the diagonals of the rectangle, and is equal to twice that straight line. 14. Use the polygon of forces. 15. Use Ex. 14: if $n$ be the number of equal parts the resultant is represented by $n$ times the radius. 16. The straight line joining $AF$ in the second diagram of Art. 58.

IV. 1. The resultant is $9\sqrt{2}$ lbs.: it is parallel to the diagonal $AC$; and it crosses $AD$ at the distance $\frac{2}{9}AD$ from $A$. 2. 38 lbs. and 114 lbs. 3. 16 inches from the heavier weight. 8. $Pa \sim Qb$. 9. $Pa2\sqrt{2}$, where $a$ is the side of the square.

V. 7. Take moments round $A$: thus we find that $KI$ is parallel to $BC$. 8. Take moments round an end of one force: thus we find it must be bisected at $O$.

VI. 1. $\sqrt{(20)}$ lbs.: see Art. 58. 5. The angle $ACB$ is given; and since $P$, $Q$, and $R$ are given, the angles which the direction of $R$ makes with $AC$ and $CB$ are given. 6. See Art. 39, and Euclid, III. 21, 22. 8. The point must be at the intersection of the straight lines which join the middle points of opposite sides. 9. The forces 1 and $\sqrt{3}$ are at right angles; the forces 2 and 1 at $120^0$. 11. Let $CD$ be the resultant of $CA$ and $CB$. Let $A$ come to $a$. Take $Dd$ equal and parallel to $Aa$; then $ad$ is equal and parallel to $CB$. Thus $Cd$ is the resultant of $Ca$ and $CB$.

VII. 1. 12 lbs. 2. 8 inches. 3. One inch from the fulcrum. 5. 2 lbs. or 5 lbs. 6. 4 lbs. 7. 3 to 4. 8. 9 to 20. 9. $26\frac{2}{3}$ cwt. 13. $P+R=Q+S$; $P=Q=R=S$. 14. See Art. 38. 15. It may be shewn that the point in the rod at which the resultant of the two weights acts is 13 inches from $C$. Then use Example 14: the tensions will be found to be 150 lbs. and 52 lbs.

VIII. 2. $P$ and $Q$. 3. A force of 12 lbs. at 5 inches from the end at which the force of 4 lbs. acts. 4. At a distance from the centre of the hexagon equal to one-fifth of a side. 5. At the point at which the force of 8 lbs. acts. 6. At the distance $\frac{1}{n-1}$ of the radius from the centre. 7. $6\frac{1}{6}$ inches from the end. 12. The force at the middle point of $BC$ must be $Q+R-P$; and so on. 17. On the diagonal through the point where no force acts, at $\frac{4}{7}$ of the diagonal from this point.

IX. 2. One foot from the end. 3. Suppose the straight line parallel to $BC$; let $D$ be the middle point of $BC$: the centre of gravity is on $AD$ at the distance $\frac{7}{9}AD$ from $A$. 4. At a distance from the centre of the larger circle equal to one-sixth of the radius. 5. Equal forces. 9. The ratio must be $\frac{1}{2}$. 13. At a distance from the

centre of the square equal to $\frac{1}{21}$ of the diagonal of the square. 14. $\frac{l}{n}$. 15. $\frac{4}{3}$, 1, $\frac{4}{5}$ feet. 16. At a distance from the base of the triangle equal to $\frac{5}{18}$ of the altitude of the triangle. 17. At a distance from the base of the triangle equal to $\frac{3}{8+2\sqrt{3}}$ of the base. 18. Put the rods so that the points in contact may be $\frac{4}{5}$ of a foot from the middle of each, towards the 1 lb. of the lower rod, and towards the 9 lbs. of the upper rod. 19. At $\frac{7}{18}$ of the whole length from the end of the densest part. 20. $8\frac{1}{3}$, $8\frac{1}{3}$, $3\frac{1}{3}$ lbs.

X. 1. Three quarters of the square. 2. A straight line parallel to the base. 3. The centre of the spherical surface. 6. One is double the other. 10. Twelve inches. 11. The distance of the point from one end of the side must be twice its distance from the other end.

XI. 1. 1 to 3. 2. 52 inches from the end. 3. $1\frac{3}{10}$ lbs., $5\frac{2}{10}$ lbs. 4. Two feet from the end. 5. Two inches. 6. 9 lbs., 6 lbs.: ratio that of 2 to 3. 7. 3 lbs. 8. 5 lbs., 7 lbs. 9. The forces are 3 lbs. and 12 lbs. 10. $12\frac{1}{2}$ lbs., $22\frac{1}{2}$ lbs. 11. One is double the other. 12. $2\frac{1}{3}$, $4\frac{1}{3}$ feet. 13. $\frac{5}{6}$ lb., $4\frac{1}{6}$ lbs. 14. 144 stone. 16. $30^0$ with Lever; $\sqrt{(12)}$ lbs. 17. $\frac{5}{3}P$ at a distance $1\frac{1}{5}$ feet from the fulcrum. 19. One inch from $A$; 10 lbs. 20. 4 inches from the fulcrum. 22. $Q=2P$.

XII. 2. $18\frac{2}{7}$ ounces. 3. 40 lbs. 4. He gets 15 ounces for 3*s.* 9*d.*; which is at the rate of 4*s.* per lb. 6. Seven inches from the point of suspension. 9. The point $D$, from which the graduations begin, is brought nearer to the point of suspension $C$. 10. The point $D$ is taken further from $C$. 11. $2\frac{1}{2}$ feet from the end at

which 10 lbs. is suspended; 60 lbs. 13. Pressure on $C$ is half the weight of the rod, on $D$ is $\frac{3}{2}$ of the weight of the rod. 14. Two feet from the other end. 15. Two lbs. 16. As $\frac{c}{2}-b$ is to $\frac{c}{2}-a$. 17. 6 feet; $\frac{1}{3}W$, $\frac{2}{3}W$. 18. 20 lbs. 20. 8 inches from the other end. 21. At $\frac{3}{14}$ of an inch from the end of the lead bar. 22. 30 lbs. 24. At a distance of $\frac{5}{12}$ of the Lever from the end where the greater force acts.

XIII. 1. 56 inches. 2. $6\frac{1}{2}$ lbs. 3. The radius of the Wheel must be 10 times the radius of the Axle. 4. 16 ounces. 5. The weight of 6 lbs. The prop must support $\frac{3}{8}$ lb., leaving $5\frac{5}{8}$ lbs. on the Wheel to balance the 15 lbs. on the Axle. The pressure on the fixed supports is $20\frac{5}{8}$ lbs. 6. 15 cwt. 7. 18 inches; 3 inches. 8. 108 lbs. 9. The string which is nailed to the Wheel hangs vertically so that its direction just touches the Axle. 10. Increased.

XIV. 1. 3 lbs. 2. One lb. 3. A force equal to a third of his weight. 4. As 12 is to 1. 5. 16 cwt. 6. 6. 7. The Weight will overcome the Power. 8. $\frac{9}{8}$ of his weight, supposing him to pull upwards, as in Art. 196; but $\frac{7}{8}$ of his weight if the Power end of the string passes over a fixed Pully so that he pulls downwards. 9. 3 lbs. 10. One lb. 11. $W=P$. 12. $W=w$. 14. $3\frac{1}{8}$ lbs. 15. 16. 16. 6. 17. The Power will overcome the Weight. 18. $7\frac{1}{2}$ cwt. 19. $\frac{13}{14}$ of his own weight. 20. Three times the Power.

XV. 1. The perpendicular from the right angle on the length. 2. $7\frac{1}{3}$ lbs. 3. 8 lbs. 4. $45^0$; as 1 is to $\sqrt{2}$.

5. $P=\frac{3}{5}W$, $R=\frac{4}{5}W$. 6. 15 lbs. 7. 140 lbs. 8. 4. 9. $3\frac{3}{4}$ lbs. 10. 1120. 11. At an inclination of $30^0$. 12. $\surd 5$ lbs. 13. 9 lbs. 14. 14 lbs.; 50 lbs. 15. $\surd 3$ lbs., $30^0$. 16. 15 lbs. 17. $60^0$. 22. 39 lbs. to hang over.

XVI. 1. $25\surd 2$ lbs. 2. 40 lbs. 3. $60^0$. 4. As 24 is to 1. 5. 48. 6. $480\pi$ lbs. 7. $n\surd 3$. 8. $\pi$ inches. 9. 2 lbs. 10. $2\pi mn$.

XVII. 1. $BL=3$ ft.; 8 lbs. 2. 3 lbs. 3. 60 lbs. 4. $A$ must now exert a force of 40 lbs. 5. The weight of $C$ is twice the weight of $B$. 6. The weights are as the lengths of the Planes on which they are placed.

XVIII. 1. 2 inches. 2. 6 inches. 3. 16 inches. 4. 5. 5. 2 feet. 6. 6. 7. 30 inches.

XIX. 1. 1. 2. Pressure $5\surd 3$ lbs.; friction 5 lbs. 3. $\surd(8^2+3^2)$ lbs.; that is $\sqrt{73}$ lbs. 5. $45^0$. 6. $75^0$. 7. 9 lbs. 9. Any force greater than 17 tons. 10. $15\frac{9}{13}$ tons. 12. $\tan\theta=\frac{a-\mu\mu' b}{(a+b)\mu}$, where $\mu$ and $\mu'$ are the coefficients of friction for the ground and wall, $a$ and $b$ the distances of the centre of gravity from the lower and upper ends; $\theta$ the inclination to the horizon.

XX. 1. 15 lbs. 2. $2P$; $60^0$, $60^0$, $45^0$. 4. $\surd(P^2+Q^2+R^2+PQ+QR+RP)$. 5. 5.

Miscellaneous Examples. 1. As 4 is to 3. 2. 25 lbs., 60 lbs. 4. 45 lbs. 5. 48 lbs., 20 lbs. 6. It is represented by $AD$. 7. It is equal to the resultant of 2 and 4 acting at right angles. 8. $75^0$, $165^0$, $120^0$. 10. 10 lbs.; bisecting the angle formed by the parts of the string. 11. On the lower peg the resultant pressure is $W$ in a vertical direction; on each of the other pegs the resultant pressure is $W\surd 3$, and the vertical component is $\frac{3W}{2}$. 18. 5 lbs. 20. $5\surd 3$ lbs. inclined at an angle of $30^0$ to the 4 lbs. component. 21. That of the sides. 22. 35 lbs., 40 lbs. 23. As 2 is to 1. 24. 21 inches.

26. 2 feet from end. 27. 8 lbs., 12 lbs. 31. $25^0$, $65^0$. 32. $3\frac{1}{3}$ lbs. 33. 54 oz., 48 oz. 34. $1\frac{1}{6}$ feet from the 4 lbs. weight. 35. 2 lbs. 37. Weight $= 2\sqrt{3} \times$ Tension. 39. $45^0$. 40. $2\frac{1}{2}$ stone. 41. $\frac{W}{2} + \frac{2Q}{5}$, $\frac{W}{2} + \frac{3Q}{5}$. 42. In the shorter cylinder at a point which divides it in the ratio of 1 to 31. 43. The point divides the rod in the ratio of 5 to 4: six points. 44. $15\frac{1}{2}$ inches from one end; shifted $1\frac{1}{8}$ inches. 45. 8 lbs. 46. 2 ounces; 4 inches from one end. 50. 9 feet from the end near the heavier boy; 6 feet from the rail. 52. $\frac{a\sqrt{3}}{20}$, where $a$ is an edge of the cube. 53. 10 lbs.; at a point $\frac{40}{4+\sqrt{3}}$ feet from the 6 lbs. end. 54. 60 lbs. 55. The pressures would now be a horizontal force equal to the Power, and a vertical force equal to the Weight. 56. 48 lbs. 57. 20 lbs. 58. $3\frac{1}{2}$ lbs. 59. 6144 lbs. 60. $28\sqrt{2}$ lbs. 61. At an angle of $30^0$ to the plane. 63. 50 lbs. inclined at $30^0$ to the plane. 64. At the centre of gravity of the weights 2 lbs., 1 lb., and 1 lb. at the angular points of the triangle. 67. It is equal to the weight of a sphere. 70. The force on the face opposite to $P$ must be $Q+R+S-2P$, and so on. 71. The centre is the point of intersection of the perpendiculars from the angles of the triangle on the opposite sides. 82. Let $r$ be the radius of the cylinder, $a$ the inclination of the rod to the horizon; then the extreme length of the rod is $2(n+2)r \sec a$. 83. Let $w$ be the weight of the upper ball, $W$ that of each lower ball, $a$ the inclination to the vertical of the straight line joining the centre of the upper ball with the centre of one of the lower balls; then the least coefficient of friction between the upper and lower ball is $\tan\frac{a}{2}$; and between the lower balls and the table is $\frac{w}{2W+w}\tan\frac{a}{2}$.

# DYNAMICS. ANSWERS.

I. 1. As 2 is to 1. 2. As 6 is to 7. 3. 15, 10. 4. $73\frac{1}{3}$. 5. As $\pi$ is to 1. 6. As $a^2$ is to $b^2$.

II. 1. 240 feet. 2. At the end of 5 seconds. 3. At the end of two minutes; at the distance 6600 feet from the starting point. 4. $5n$ feet. 5. $1527\frac{7}{9}$ feet per second. 6. $7\pi$ feet per second. 7. The distance between them is equal to the distance each has described. 8. $n\sqrt{(u^2+v^2-2uv\cos a)}$.

III. 1. 50. 2. 2 seconds. 3. 18. 4. 20; 1. 5. $25; \frac{1}{2}$. 6. As $b^2-a^2$ is to $v^2-u^2$. 7. 32. 8. 32; the first second is the third from rest. 9. The first second is the $\frac{b+a}{2(b-a)}$ th from rest. 10. $6\frac{2}{5}$. 11. 32. 12. $\frac{9}{2}g$. 13. $\sqrt{\frac{2h}{g}} - \sqrt{\frac{2h'}{g}}$; where $h$ and $h'$ are the heights. 14. 48 inches; 8 feet. 16. $2\frac{1}{2}$ seconds. 17. The radius to the point is inclined at $60^0$ to the radius which is vertically upwards. 18. The radius to the point is inclined at $60^0$ to the radius which is vertically downwards. 24. $f \cdot \frac{n}{\nu} \cdot \frac{\mu^2}{m^2}$.

IV. 1. 2 or 4 seconds; respective velocities $g$ and $-g$. 2. 3 seconds. 3. Yes, $f=32$. 4. 36; 16. 6. 5 or 20 seconds. 7. $\sqrt{(gh)}$, where $h$ is the given height. 8. It is half the time. 9. 2 seconds. 10. 2 or 18 seconds. 11. Let $u$ be the initial velocity, $h$ the height of the given point, $n$ the number of seconds between passing this point and coming to it again; then $\frac{2\sqrt{(u^2-2gh)}}{g}=n$. 13. The time is $\frac{1}{2}\sqrt{\frac{l}{2g\sin a}}$, where $l$ is the length of the wire, and $a$ its inclination to the horizon. One ring describes $\frac{9l}{16}$, and the other $\frac{7l}{16}$. 14. Let $\tau$ denote each interval; then the space is

$nu\tau + \frac{n(n-1)v\tau}{2}$; put $n\tau = t$, and $nv = ft$; eliminate $\tau$ and $v$, and finally suppose $n$ infinite.

V. 1. 13 feet per second. 2. $\frac{9g}{2}$. 3. No. 5. $\frac{2}{g} v \sin a (u + v \cos a)$. 6. Let $u$ and $v$ be the velocities of projection, and $a$ and $\beta$ the angles of projection. The square of the distance at the time $t$ is $(u \sin a - v \sin \beta)^2 t^2 + (u \cos a - v \cos \beta)^2 t^2$, that is $\{u^2 + v^2 - 2uv \cos (a - \beta)\} t^2$. 7. $\frac{u \sin a}{g} + \frac{u \cos a}{g}$. 11. The square of the distance is $t^2 u^2 \cos^2 a + \left(tu \sin a - \frac{1}{2} gt^2\right)^2$, that is $t^2 \left(u^2 - gtu \sin a + \frac{g^2 t^2}{4}\right)$. 13. Suppose the first body projected with the velocity $u$ at an inclination $a$; then the second body must be projected vertically with the velocity $u \sin a$. 14. $\dfrac{u \sin a - \frac{1}{2} gt}{u \cos a} = \tan \beta$. 15. $\frac{2u^2}{g} \cos^2 a \tan \beta$. 16. Let $u$ be the velocity with which the body is projected horizontally; then the distance at the end of the time $t$ is $ut$; and $t = \sqrt{\dfrac{2s}{g \sin a}}$, where $s$ is the given space, and $a$ the inclination of the plane to the horizon.

VI. 1. The point must be the vertex of the parabolic path, so that the values of $\frac{u^2 \sin 2a}{2g}$ and $\frac{u^2 \sin^2 a}{2g}$ are known; see Arts. 56 and 57. 2. See Art. 71. 3. From Arts. 57 and 65 we have $\frac{u \sin a}{g} = \frac{2u \sin (a - \beta)}{g \cos \beta}$ or $\tan a = 2 \tan \beta$. 8. $\frac{u \sin a - gt}{u \cos a} = \pm \cot \beta$. 9. We must now take the lower sign of the preceding result: thus we get $t = \frac{u \cos (a - \beta)}{g \sin \beta}$; and another value of $t$ is found from

Art. 65. Hence we get $2\tan(\alpha-\beta)=\cot\beta$; this gives $\tan\alpha=\dfrac{1+\sin^2\beta}{\sin\beta\cos\beta}$. 10. Let $x$ be the horizontal space described, and $y$ the vertical space. Then $x=tu\cos\alpha$, $y=x\tan\beta=tu\tan\beta\cos\alpha=\dfrac{2u^2\sin\beta\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}$ $=\dfrac{2u^2\sin\beta\cos^2\alpha}{g\cos^2\beta}(\tan\alpha\cos\beta-\sin\beta)$. This reduces to $\dfrac{2u^2\cos^2\alpha}{g\cos^2\beta}$, and this to $\dfrac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$. 12. This amounts to the fact that the time of describing a space $l$ with a velocity $V$ is the same as the time of describing a space $l\cos\beta$ with a velocity $V\cos\beta$. 16. See Art. 70. 17. $\dfrac{2}{g}v\sin\alpha\sqrt{(u^2+v^2\cos^2\alpha+2uv\cos\alpha\cos\beta)}$. 19. $\dfrac{u\sin\alpha}{g}\sqrt{2}$. 20. $x^2-4nxh\sin\alpha\cos\alpha-4nh^2\cos^2\alpha=0$. 21. From the preceding result obtain a quadratic in $\tan\alpha$; solve it, and examine the expression under the radical sign. 22. Let $t$ be the time between just passing the cube and reaching the highest point, or between reaching the highest point, and just passing the cube again; then

$$\frac{1}{2}gt^2=\frac{u^2\sin^2\alpha}{2g}-c, \quad ut\cos\alpha=\frac{c}{2},$$

therefore $4u^4\sin^2\alpha\cos^2\alpha-8cgu^2\cos^2\alpha-c^2g^2=0$.
23. $c^2g^2\tan^4\alpha+\tan^2\alpha(8cgu^2+2c^2g^2-4u^4)+c^2g^2+8cgu^2=0$. Solving the quadratic for $\tan^2\alpha$, we find that under the radical sign we have the expression $u^4-4cgu^2+3c^2g^2$, that is $(u^2-3cg)(u^2-cg)$. From this we infer that $u^2$ must be greater than $3cg$, for it cannot be less than $cg$, as we see from the first formula in Ex. 22. 25. See Ex. 13.

VII. 1. $\dfrac{na}{g}$ lbs. 2. $\dfrac{4}{9}g\dfrac{t^2}{2}=49$; $v=\dfrac{4}{9}gt$. 3. $s=\dfrac{n}{m}\dfrac{gt^2}{2}$, $v=\dfrac{n}{m}gt$. 4. $\dfrac{1}{2}\dfrac{g}{16}\left(\dfrac{1}{2}\right)^2 12$. 5. $m'=4m$, $v'=\dfrac{1}{2}v$, $m'v'=2mv$. 6. Let the pressure be $p$ lbs.: then $\dfrac{8-p}{8}=\dfrac{12}{g}$. 7. Let the pressure be $p$ lbs.: then $\dfrac{p-n}{n}=\dfrac{f}{g}$. 8. Take the unit of mass, then $M=1$ and $W=1$; thus $g=1$. Let

the unit of time be $t$ seconds; then as $\frac{1}{2}g$ is the space through which a body falls in a unit of time, we must take the unit of time such that a body should fall through 1 foot during it. Let $t$ be the number of seconds, then $\frac{t^2}{2} \cdot 32=1$; thus $t=\frac{1}{4}$. 9. 9 feet. 10. $g(\sqrt{3}+1)$. 12. If $r$ denote the length of a plane inclined at an angle $\theta$ to the horizon, we find that $r(\sin\theta-\mu\cos\theta)$ must be constant; that is $r\sin(\theta-\epsilon)$ must be constant, where $\mu=\tan\epsilon$. Thus the starting point must be at a constant distance from a straight line drawn through the origin which makes an angle $\epsilon$ with the horizon. Two such straight lines can be drawn; and the required locus is two straight lines parallel to these respectively.

VIII. 1. $s=25$, $v=10$. 2. $2ut$ added to the distance at the time of cutting. 4. $\frac{Qg}{P+Q}$. 5. $Q=P$. 7. Three on one side of the Pully, and one on the other side. 8. Through $\frac{3}{4}$ of the given space. 9. $\frac{9}{8}$ lbs., $\frac{21}{8}$ lbs. 10. $\frac{m^2-m'^2}{m'}g$.

IX. 2. 11, 13. 3. $\frac{1}{2}$; $m'=2m$. 7. $m'=em$. 9. $\frac{AC+B^2}{B(A+C)}$. 12. $B$'s mass $=\frac{1}{e}$ times $A$'s mass; and so on; $e^{n-1}u$ where $u$ is the original velocity of $A$.

X. 1. $45^0$. 2. $e^2h$. 3. $h+\frac{2e^2h}{1-e^2}$. 5. $\tan^2\alpha=\frac{m'e-m}{m'+m}$. 6. $u\sin 30^0$. 7. $\alpha=45^0$. 8. $e^n\tan\alpha$. 9. $\frac{2u^2\sin\alpha\cos\alpha}{g(1-e)}$. 10. $\frac{2u\sin\alpha}{g(1-e)}$. 12. Let $b$ be the length of the *adjacent* side; the ball must hit this side at the distance $\frac{b}{1+e}$ from the

end nearest to the *opposite* side. 14. $\tan^2 a = e$. 16. The angle $AFD$ must be $90^0$, and the angle $DFE$ must be $135^0$. 18. $4eh \sin a \cos a$ where $h$ is the height of the plane, and $a$ its inclination to the horizon. 19. Let $c$ be the distance of the wall from the point of projection; then the time of motion $= \frac{2v \sin a}{g}$ and $= \frac{c}{v \cos a} + \frac{c}{ev \cos a}$; therefore $v^2 \sin 2a = gc\left(1 + \frac{1}{e}\right)$. 23. At the foot of the first wall.

XI. 1. 4 feet per second. 2. 5 feet per second. 3. $\left(\frac{m - m'}{m + m'}\right)^2 gt$. 4. Let $m$ be the mass of the body hanging over the plane, $m'$ that of the other: then at the end of the time $t$, the vertical velocity of the centre of gravity is $\frac{m^2 gt}{(m + m')^2}$, and the horizontal velocity is $\frac{mm'gt}{(m + m')^2}$. 6. The velocity of the centre of gravity is composed of $\frac{m(m \sin a - m' \sin a')gt}{(m + m')^2}$ parallel to the plane on which is the body of mass $m$, *downwards*, and $\frac{m'(m \sin a - m' \sin a')gt}{(m + m')^2}$ parallel to the other plane *upwards*.

XII. 2. The square of the distance at the time $t$ will be found to be $V^2 t^2 - 2uta + a^2$, that is $\left(Vt - \frac{ua}{V}\right)^2 + \frac{v^2 a^2}{V^2}$: hence the distance is least at the end of the time $\frac{ua}{V^2}$. 3. $u \cos a - u' \cos a' + (f \cos a - f' \cos a')t$.

XIII. 1. $\sqrt{rg}$. 3. $\left(\frac{8652}{8640}\right)^2$. 6. $\pi \sqrt{\frac{l}{g \sin a}}$, where $a$ is the inclination of the plane to the horizon; and $\pi \sqrt{\frac{l}{g}} = 1$. 7. $\frac{4000 + m}{4000}$ seconds.

XIV. 1. Acceleration $\frac{4\pi^2}{5}$. 2. $2\pi \sqrt{\frac{2}{g}}$ seconds.

3. The point is the centre of gravity of the $P$ lbs. and the $Q$ lbs. 4. $\sqrt{\left(\frac{Plg}{Q}\right)}$.

XV. 1. 8 years. 2. $\frac{1}{(60)^{\frac{3}{2}}}$ of the moon's period.
3. As $1+e$ is to $1-e$, where $e$ is the excentricity.

XVI. 1. $\frac{2\sqrt{2}}{3}$. 2. $\pi$ seconds.

XVII. 1. 67200. 2. 2640. 3. 3510. 4. 64.
5. 110·08. 6. $112000 \times \pi$; for the centre of gravity of the part removed is at the depth of 10 feet. 7. 499·2.
8. ·65. 9. 19200. 10. 5·2 nearly. 11. 25.
12. 7 days, $19\frac{1}{2}$ days. 13. 9. 14. 346·4.
15. $\frac{m \times 20 \times 112}{2g}(v^2 - u^2) + mns$. 16. $\frac{1000abche}{16 \times 60m}$.
17. $\frac{a}{2}$. 18. 1320 feet per minute. 19. 7·233 nearly.
20. 29120 nearly. 21. 2·3 nearly. 22. 508 nearly.
23. 7500000. 24. About 990.

XIX. 7. $\frac{M}{36}(a^2+b^2+c^2)$. 8. $\frac{M}{12}(a^2+b^2)$ about the axis parallel to the edge $c$.

XX. 1. $\frac{2a}{3}$. 2. $\frac{2\sqrt{2}}{3}a$. 3. $\frac{5\sqrt{3}}{12}a$.

MISCELLANEOUS. 1. $2\frac{1}{2}$ seconds. 2. 144 feet. 3. 9·2.
8. A horizontal straight line. 9. $g$. 12. This may be deduced from the geometrical fact that the two tangents to a parabola from any point in the directrix are at right angles. 13. See Art. 70; the difference of the distances of the focus from the two fixed points is constant.
14. $27\frac{1}{2}$. 15. Let $T$ be the tension of the string, $m$ the mass of $P$, and $m'$ the mass of $Q$; the acceleration of $P$ is $\frac{mg-T}{m}$ downwards, and that of $Q$ is $\frac{2T-m'g}{m'}$ upwards; and at any instant $P$ is moving downwards with twice

the velocity with which $Q$ is moving upwards: thus $\frac{mg-T}{m}=2\times\frac{2T-m'g}{m'}$; thus $T=\frac{3mm'g}{4m+m'}$. 17. If $m=em'$ then $v=u'$, and $v'=eu+(1-e)u'$; if $m'=em$ then $v'=u$, and $v=(1-e)u+eu'$. 18. The blow must communicate a momentum $\sqrt{2}$ times that which the body has, in a direction making an angle of $135^0$ with that of the original motion. 19. $v\sqrt{(\sin^2 a+\cos^2 a\cos^2\beta+e^2\cos^2 a\sin^2\beta)}$.
20. $\frac{u}{2^5}$, where $u$ is the original velocity. 21. $\frac{u\sqrt{3}}{2}, \frac{u\sqrt{3}}{4}, \frac{u\sqrt{3}}{8}, \frac{u\sqrt{3}}{16}, \frac{u\sqrt{3}}{32}$. 23. In the old directrix.
24. We get two expressions for the whole time, namely $\frac{2}{g}\left(1+\frac{1}{e}\right)(u\sin a-u\cos a\tan\beta)$ and $\frac{2u\sin a}{g}$: equate them.
25. $\frac{2u\sin(a-\beta)}{g\cos\beta}\cdot\frac{1-e^n}{1-e}$. 26. $\frac{\sin(a-\beta)}{\cos\beta}\cdot\frac{1-e^n}{1-e}=\frac{\cos(a-\beta)}{\sin\beta}$.
27. $\frac{2u\sin(a-\beta)}{g(1-e)\cos\beta}$. 28. $\tan(a-\beta)\tan\beta=1-e$.
29. $\frac{\cot(a-\beta)}{e^n}-\frac{2(1-e^n)\tan\beta}{e^n(1-e)}$.
30. $\cot(a-\beta)=(2n+1)\tan\beta$.
32. The velocity varies as the sixth root of the tonnage.
33. $\left(\frac{1+e}{2}\right)^2 h$, $\left(\frac{1-e}{2}\right)^2 h$. 34. $\frac{(n+1)m}{n(mn+1)}$.

THE END.

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